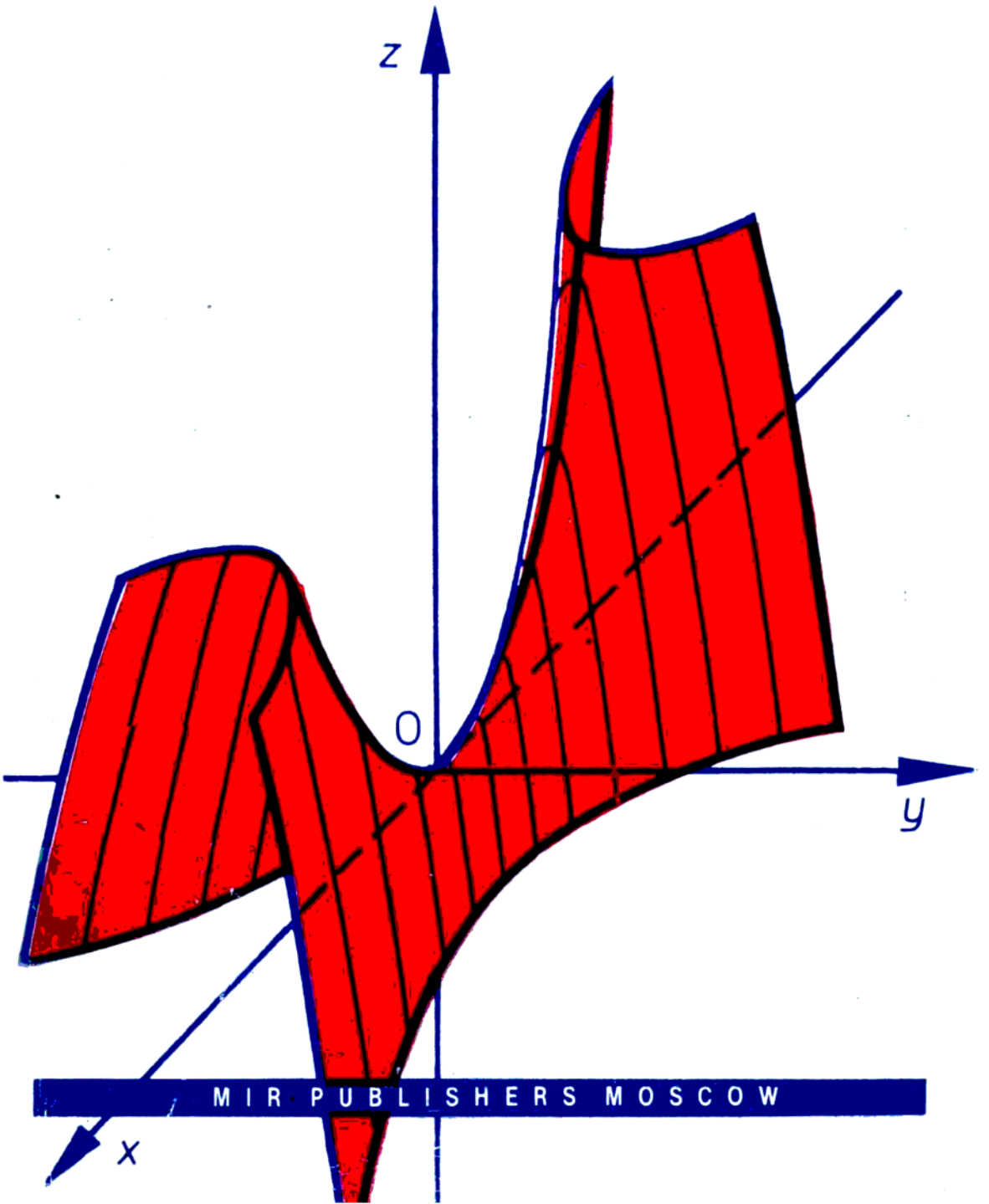
HIGHER MATHEMATICS

EDITED BY G. N. YAKOVLEV



MIR PUBLISHERS MOSCOW

HIGHER MATHEMATICS

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Edited by G. N. Yakovlev,
Corr. Member of the USSR Academy
of Pedagogical Sciences

MIR PUBLISHERS MOSCOW

Translated from the Russian
by Leonid Levant

First published 1990
Revised from the 1988 Russian edition

На английском языке

Printed in the Union of Soviet Socialist Republics

ISBN 5-03-001570-1
ISBN 5-09-000293-2

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PREFACE TO THE ENGLISH EDITION

Mathematical methods are used in every field of science and engineering. This means that students, whatever their speciality, must have a solid theoretical grounding in mathematics and in solving practical problems. Mathematical analysis is the foundation of the mathematical education of scientists and engineers.

This study aid was written in accordance with the new curriculum on higher mathematics and is intended for students of the industrial and education faculties of teacher training institutes. Attention is given to the applied and practical approach of the text and the application of computers.

The book is in two parts: Part 1. Analytic Geometry (Chapters 1-3) and Part 2. Mathematical Analysis (Chapters 4-12).

Chapter 1 provides all necessary theoretical elements of vector and linear algebra. Chapter 2 deals with analytic geometry in the plane, while Chapter 3 with analytic geometry in space. Chapter 4 treats real and complex numbers. Chapter 5 is dedicated to functions, sequences, and limits. Elements of differential calculus are discussed in Chapter 6 (functions of one variable) and Chapter 10 (functions of several variables). Chapters 7 and 8 deal with indefinite and definite integrals, respectively. Series are considered in Chapter 9. Chapter 11 provides a study of differential equations, and Chapter 12 is dedicated to the integral calculus of functions of several variables.

Each chapter contains a large number of examples and worked problems to make the subject matter more readily comprehensible and is amply supplied with exercises.

As a rule, worked examples in each section are arranged in order of increasing difficulty. The beginning and end of the proof of a statement are marked by the symbols $\square$ and $\blacksquare$, respectively. The solution of each example is preceded by the symbol $\triangle$, the end of the solution being symbolized by $\blacktriangle$.

The Authors

Part I

ANALYTIC GEOMETRY

Chapter 1

ELEMENTS OF VECTOR AND LINEAR ALGEBRA

1.1. Vectors. Operations on Vectors

1. Scalar and Vector Quantities. It is known that some quantities are completely defined by their numerical value (with a certain unit of measurement chosen). Such quantities are called scalar (or numerical). Length, area, volume, mass, and temperature are examples of scalar quantities. Other quantities, as, for instance, displacement, force, acceleration, and velocity, are defined not only by their numerical value, but also by direction in space. These are called vector quantities, or simply vectors. Returning to scalar quantities, we may give the following definition: a scalar quantity (or scalar) is a quantity that does not possess direction.

Geometrically, a vector quantity (a vector) can be represented with the help of a directed line segment whose length in a given scale of measurement is equal to the numerical value of the vector quantity and whose direction coincides with the direction of this quantity. A directed line segment is represented by an arrow. Figure 1 shows a directed line segment with initial point (or origin) O and terminal point A. It is denoted as $\overrightarrow{OA}$. Any other directed line segment is denoted in a similar way, provided that its beginning and terminus are known. For example, $\overrightarrow{AB}$ denotes a directed line segment with initial point A and terminal point B. The length of the directed line segment $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

Thus, a vector is a quantity which can be represented by a directed line segment. This directed line segment is called the vector representing a given vector quantity. Thus, in geometry, a vector is a directed line segment.

Definition 1. Two vectors are equal if they are of the same length, lie on parallel or coinciding lines, and are in the same direction.

For example, in Fig. 2, where $ABCD$ is a rhombus, the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are equal since $|\overrightarrow{AD}| = |\overrightarrow{BC}|$, $AD \parallel BC$ and the segments $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are in the same direction. In this case we write: $\overrightarrow{AD} = \overrightarrow{BC}$.

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in Fig. 2 are not equal to each other since the straight lines AB and AD are not parallel. In this case we write: $\overrightarrow{AB} \neq \overrightarrow{AD}$. Further, since $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are in opposite directions, we have $\overrightarrow{AB} \neq \overrightarrow{CD}$.

A vector can also be denoted by a single letter as in Fig. 4. In printing this letter is given in boldface type **a**, in writing it is given with

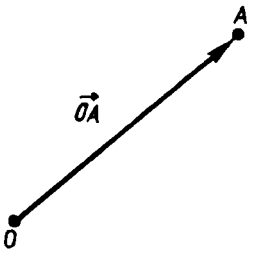

Fig. 1

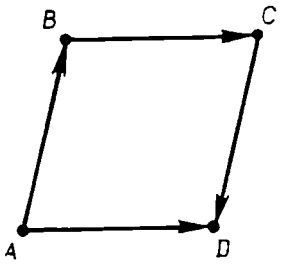

Fig. 2

a bar $\bar{a}$. If it is known that the vector **a** begins at the point A and terminates at the point B, then we write: $\mathbf{a} = \overrightarrow{AB}$. The length of the vector **a** is denoted by two vertical lines: $|\mathbf{a}|$. Sometimes, the length of a vector is called the absolute value.

A vector whose length is equal to zero is called the null vector. The null vector is symbolized by **0**, the number zero. The origin A of a null vector coincides with its terminus B, that is, $\mathbf{0} = \overrightarrow{AA} = \overrightarrow{BB}$.

Every nonzero vector is defined by length and direction. The null vector has no direction.

Definition 2. Two nonzero vectors whose directions coincide or are opposite are termed collinear. The null vector is considered collinear with any vector.

If two vectors **a** and **b** are collinear, then we write: $\mathbf{a} \parallel \mathbf{b}$. It is obvious that collinear vectors are those lying on parallel or coinciding straight lines. In Fig. 2, $\overrightarrow{AD} \parallel \overrightarrow{BC}$, $\overrightarrow{AB} \parallel \overrightarrow{CD}$, but $\overrightarrow{AB} \nparallel \overrightarrow{BC}$.

Definition 3. Two collinear vectors having the same absolute value (length) and opposite directions are called opposite vectors.

A vector which is in the direction opposite to a vector **a** is denoted by $-\mathbf{a}$. It is obvious that the vector $\overrightarrow{BA}$ is opposite to the vector $\overrightarrow{AB}$, and vice versa, that is, $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{AB} = -\overrightarrow{BA}$. In Fig. 2, $\overrightarrow{AB} = -\overrightarrow{CD}$.

Definition 4. Vectors in space are called coplanar if they are parallel to the same plane.

It is obvious that any two vectors in space are coplanar. In Fig. 3, where $ABCDEFGH$ is a parallelepiped, the vectors $\vec{AB}$, $\vec{BC}$, $\vec{CD}$, $\vec{AD}$, $\vec{FE}$, $\vec{FG}$, $\vec{HG}$, and $\vec{HE}$ are coplanar.

2. The Angle Between Two Vectors. The Projection of a Vector on an Axis. Suppose we are given two vectors **a** and **b**. We lay off these vectors from a certain point O, that is, we construct vectors $\vec{OA}$ and $\vec{OB}$ such that $\mathbf{a} = \vec{OA}$ and $\mathbf{b} = \vec{OB}$ (Fig. 4). Then the size of the interior angle AOB of the triangle AOB is defined as the angle between the vectors **a** and **b** and is denoted by $(\widehat{\mathbf{a}, \mathbf{b}})$ or by letters φ, ψ, etc. By definition, the angle between two vectors lies in the range

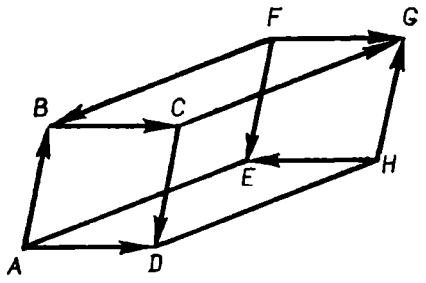

Fig. 3

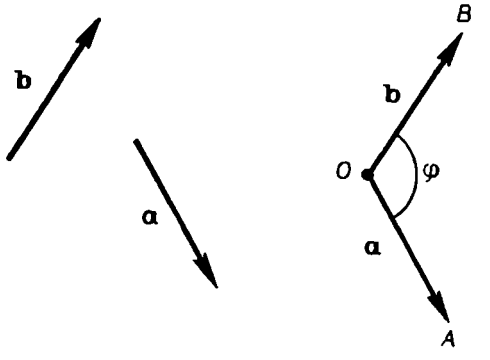

Fig. 4

from 0° to 180° (or from 0 to π radians). The angle between collinear vectors is equal to zero if they are in the same direction, and to 180° if they are in opposite directions.

If the angle between vectors is 90°, then these vectors are called perpendicular or orthogonal. If the vectors **a** and **b** are perpendicular, then we write: $\mathbf{a} \perp \mathbf{b}$.

Recall that an axis is defined as a straight line with a reference point, positive direction, and a length unit (scale) chosen. It is obvious that positive direction and unit of length on the axis can be specified with the help of a unit vector, that is, the vector whose length is equal to the unit of length. This vector is termed the unit vector of the axis. Figure 5 represents the axis l, where O is the reference point, and **e** the unit vector. The angle between the vector **a** and the axis l is defined as the angle between the vectors **a** and **e**.

Definition. The orthogonal projection (or simply projection) of a vector on an axis is a number equal to the product of the length of this vector by the cosine of the angle between the vector and axis.

The projection of the vector **a** on the axis l is denoted by the sym-

bol $\mathrm{pr}_l \mathbf{a}$. Thus, by definition

$$\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cos \varphi, \tag{1}$$

where φ is the angle between the vector $\mathbf{a}$ and the axis l. If the angle between the vectors $\mathbf{a}$ and $\mathbf{e}$ is acute (Fig. 5, *a*), then the projec-

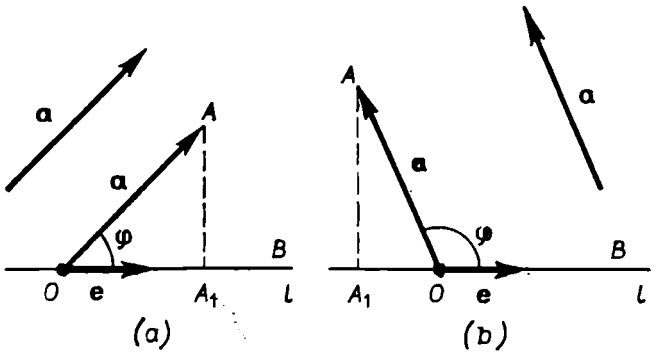

Fig. 5

tion of the vector $\mathbf{a}$ on the axis l is equal to the length of the line segment OA_1, where the point A_1 is the projection of the point A on the straight line l. Indeed,

$$\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cos \varphi = |\overrightarrow{OA}| \cos \angle AOA_1 = OA_1.$$

If the angle between the vectors $\mathbf{e}$ and $\mathbf{a}$ is obtuse (as in Fig. 5, *b*), then the projection of the vector $\mathbf{a}$ on the axis l is equal to the length of the line segment OA_1 taken with the minus sign. Indeed,

$$\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cos \varphi = |\mathbf{a}| \cos \angle BOA = -|\overrightarrow{OA}| \cos \angle A_1OA = -OA_1.$$

If the vector $\mathbf{a}$ is perpendicular to the axis l, that is, $\varphi = 90°$, then $\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cos 90° = 0$.

Example. Compute the projection of the vector $\mathbf{a}$ on the axis l if $|\mathbf{a}| = 6$, and the angle φ between the vector and the axis is equal to $\frac{2\pi}{3}$.

△ We compute by the formula (1): $\mathrm{pr}_l \mathbf{a} = |\mathbf{a}| \cos \varphi = 6 \cos \frac{2\pi}{3} = 6 \cos \left(\frac{\pi}{2} + \frac{\pi}{6}\right) = -6 \sin \frac{\pi}{6} = -3$. ▲

3. The Sum of Vectors. Let a body move from the point A to the point B, and then from the point B to the point C (Fig. 6). In mechanics the vector $\overrightarrow{AB}$ is called the displacement of the body from the point A to the point B. Similarly, the vector $\overrightarrow{BC}$ is the displacement

of the body from the point B to the point C, while the vector $\overrightarrow{AC}$ is the displacement from the point A to the point C. The vector $\overrightarrow{AC}$ is then the sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. In this case we write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Let us give the general definition for the sum of two vectors.

Definition. Let two vectors **a** and **b** be given. We lay off the vector **a** from some point A: $\mathbf{a} = \overrightarrow{AB}$, and then the vector **b** from the

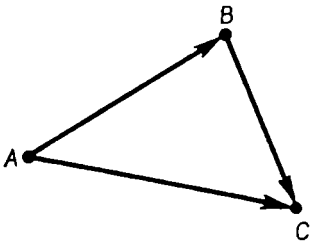

Fig. 6

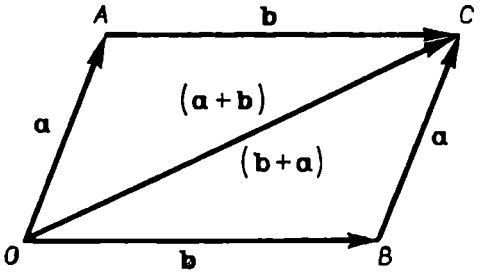

Fig. 7

point B: $\mathbf{b} = \overrightarrow{BC}$. Then the vector $\mathbf{c} = \overrightarrow{AC}$ is said to be the sum of the vectors **a** and **b**. In this case we write:

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

The operation of finding the sum of vectors is called the addition of vectors. The above formulated rule for adding vectors is referred to as the triangle rule.

The operation of addition of vectors possesses the properties of commutativity and associativity, that is, for any vectors **a**, **b**, and **c** the following equalities hold true:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \text{ (commutativity)},$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \text{ (associativity)}.$$

□ Let us first prove the commutative property. We lay off the vectors **a** and **b** from some point O: $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$. Consider the case when the vectors **a** and **b** are noncollinear, that is, when the points O, A and B do not lie on the same straight line. On the line segments OA and OB we complete a parallelogram $OBCA$ (Fig. 7), then

$$\mathbf{a} + \mathbf{b} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC},$$

$$\mathbf{b} + \mathbf{a} = \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC},$$

and, consequently, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

For the case when the vectors **a** and **b** are collinear, the proof is left to the student.

Let us now prove the associative property. We lay off the vector $\overrightarrow{OA} = \mathbf{a}$ from some point O, the vector $\overrightarrow{AB} = \mathbf{b}$ from the point A, finally the vector $\overrightarrow{BC} = \mathbf{c}$ from the point B (Fig. 8). Then

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}=\overrightarrow{OB}+\overrightarrow{BC}=\overrightarrow{OC},$$

$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC},$$

and consequently, $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$. ■

Since the addition of vectors is associative, the sum of three and more vectors is written without parentheses. For instance, instead

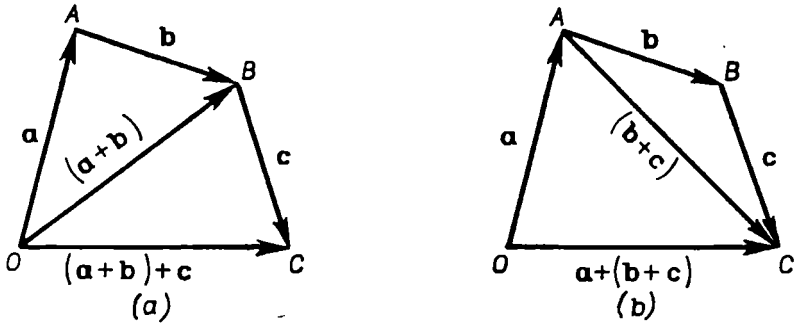

Fig. 8

of $(\mathbf{a}+\mathbf{b})+\mathbf{c}$ or $\mathbf{a}+(\mathbf{b}+\mathbf{c})$, we write: $\mathbf{a}+\mathbf{b}+\mathbf{c}$. If it is required to find the sum of three or more vectors, then we apply the so-called polygon rule (for the three-dimensional case, it would be more exact to call this rule the polygonal-line rule), which is the consequence of the triangle rule for adding two vectors. It consists in the following. Let us be given vectors **a**, **b**, **c**, **d**, etc. We choose some point O and construct the line segment $\overrightarrow{OA} = \mathbf{a}$, then the line segment $\overrightarrow{AB} = \mathbf{b}$, and so forth until all the vectors to be added are exhausted. The directed line segment emanating from the first vector and terminating at the terminus of the last vector thus closing the polygonal line obtained in the process of the described construction will be equal to the sum of the given vectors.

Example 1. Given a parallelepiped $ABCDEFGH$ (see Fig. 3). Find the sum of the vectors $\overrightarrow{AB}$, $\overrightarrow{FG}$, $\overrightarrow{CG}$, $\overrightarrow{FE}$, $\overrightarrow{FB}$.

△ From the properties of the edges of the parallelepiped it follows that $\overrightarrow{FG}=\overrightarrow{BC}$, $\overrightarrow{FE}=\overrightarrow{GH}$, $\overrightarrow{FB}=\overrightarrow{HD}$. Therefore $\overrightarrow{AB}+\overrightarrow{FG}+$

$\overrightarrow{CG}+\overrightarrow{FE}+\overrightarrow{FB}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CG}+\overrightarrow{GH}+\overrightarrow{HD}=\overrightarrow{AD}$, where we have applied the polygonal-line rule. ▲

Example 2. Given a triangular pyramid $ABCD$ (Fig. 9). Find the sum $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{AC}+\overrightarrow{BC}+\overrightarrow{DA}$.

△ Applying the commutative and associative properties of the addition of vectors, we get $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{AC}+\overrightarrow{BC}+\overrightarrow{DA}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}+\overrightarrow{AC}=\mathbf{0}+\overrightarrow{AC}+\overrightarrow{AC}$. ▲

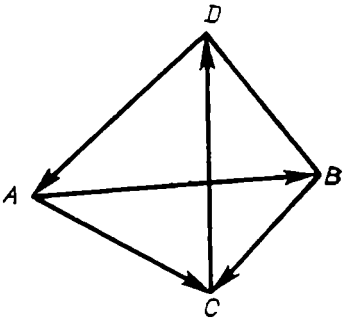

Fig. 9

It is obvious that for any vectors **b** and **c** the following equality is valid:

$$\mathrm{pr}_l\,(\mathbf{b}+\mathbf{c})=\mathrm{pr}_l\,\mathbf{b}+\mathrm{pr}_l\,\mathbf{c}. \qquad (1)$$

Indeed, if the projections of the vectors **b** and **c** on some axis l are of the same sign (Fig. 10, *a*), then the following equalities are valid:

$$\mathrm{pr}_l\,(\mathbf{b}+\mathbf{c})=\mathrm{pr}_l\,\overrightarrow{AC}=A_1C_1$$
$$=A_1B_1+B_1C_1=\mathrm{pr}_l\,\mathbf{b}+\mathrm{pr}_l\,\mathbf{c},$$

and if the projections are of opposite sign (Fig. 10, *b*), then we have:

$$\mathrm{pr}_l\,(\mathbf{b}+\mathbf{c})\;\mathrm{pr}_l\,\overrightarrow{AC}=-A_1C_1=A_1B_1-B_1C_1=\mathrm{pr}_l\,\mathbf{b}+\mathrm{pr}_l\,\mathbf{c}.$$

In both cases equality (1) is valid.

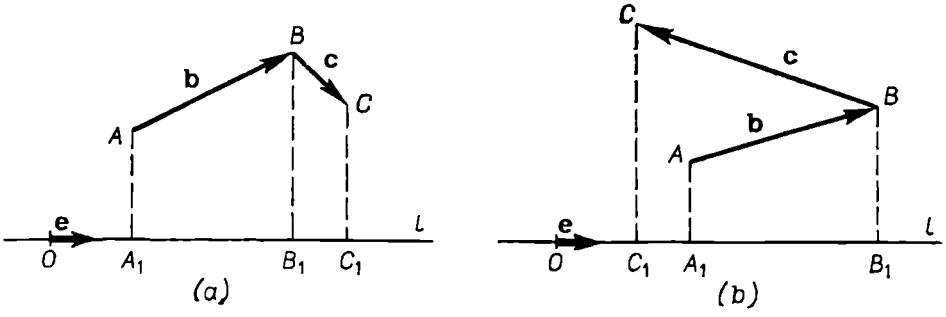

Fig. 10

4. The Difference Between Two Vectors. Recall that for any vector **a** its opposite vector is denoted by $-\mathbf{a}$. Obviously, $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$.

Definition. For any vectors **a** and **b** the sum $\mathbf{a}+(-\mathbf{b})$ is called their difference and is denoted as $\mathbf{a}-\mathbf{b}$.

Consequently, by definition, $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$.

Let us give the rule for finding the difference between two vectors. Given vectors **a** and **b** (Fig. 11). We lay off the vector **a** from some

point O: $\mathbf{a} = \overrightarrow{OA}$, and the vector $(-\mathbf{b})$ from the point A: $-\mathbf{b} = \overrightarrow{AB}$. Then $\mathbf{a} - \mathbf{b} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

5. The Product of a Vector by a Number. For vectors, besides the operations of addition and subtraction, the operation of multiplication of a vector by a number is also defined.

Definition. The product of a nonzero vector $\mathbf{a}$ by a number $\lambda \neq 0$ is a vector whose length is equal to $|\lambda|\,|\mathbf{a}|$, the direction coinciding

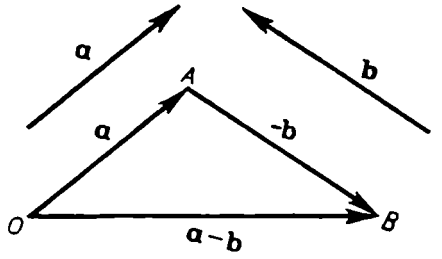

Fig. 11

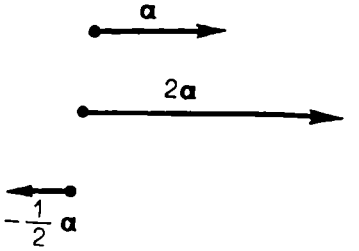

Fig. 12

with the direction of the vector $\mathbf{a}$ or being in the opposite sense, depending on whether the number is positive or negative. If $\lambda = 0$, the product is the null vector.

The product of the vector $\mathbf{a}$ by the number λ is denoted by $\lambda\mathbf{a}$ (the numerical factor is usually written first). The product of the null vector by any number λ, as well as the product of any vector by the number zero, is the null vector. By definition

$$|\lambda\mathbf{a}| = |\lambda|\,|\mathbf{a}|,$$

where $|\lambda|$ is the absolute value of λ. Figure 12 shows the vectors $\mathbf{a}$, $2\mathbf{a}$, and $-\frac{1}{2}\mathbf{a}$.

The operation of multiplication of a vector by a number possesses the associative and distributive properties, that is, for any vectors $\mathbf{a}$ and $\mathbf{b}$ and any numbers λ, μ the following equalities hold:

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a}, \tag{1}$$

$$\lambda\mathbf{a} + \mu\mathbf{a} = (\lambda + \mu)\,\mathbf{a}, \tag{2}$$

$$\lambda\mathbf{a} + \lambda\mathbf{b} = \lambda(\mathbf{a} + \mathbf{b}). \tag{3}$$

□ Let us prove equality (1). If $\mathbf{a} = 0$ or $\lambda\mu = 0$, then the equality $\lambda(\mu\mathbf{a}) = (\lambda\mu)\,\mathbf{a} = 0$ is obvious. Let $\mathbf{a} \neq 0$, $\lambda\mu \neq 0$, and $\mathbf{a} = \overrightarrow{OA}$. Then the vectors $\lambda(\mu\overrightarrow{OA})$ and $(\lambda\mu)\,\overrightarrow{OA}$ lie on the straight line $\overrightarrow{OA}$, have the same length equal to $|\lambda|\,|\mu|\,|\overrightarrow{OA}|$, and are in the same direction (in the direction of $\overrightarrow{OA}$ if $\lambda\mu > 0$, and in the op-

posite direction if $\lambda\mu < 0$). Hence, by definition, they are equal. Equalities (2) and (3) are left to the student. ■

Example 1. In the rhombus $ABCD$, N is the point of intersection of the diagonals. Find the factor λ in the following equalities: (a) $\overrightarrow{NC} = \lambda\overrightarrow{CA}$; (b) $\overrightarrow{BD} = \lambda\overrightarrow{BN}$; (c) $\overrightarrow{AC} = \lambda\overrightarrow{CN}$; (d) $\overrightarrow{BB} = \lambda\overrightarrow{BD}$ (Fig. 13).

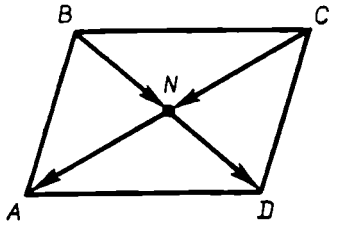

Fig. 13

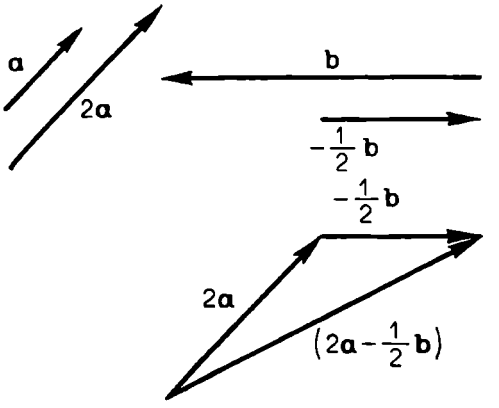

Fig. 14

△ Using the definition of the multiplication of a vector by a number and bearing in mind the properties of the diagonals of the rhombus we have:

(a) $\overrightarrow{CA}$ and $\overrightarrow{NC}$ are in opposite directions, $|\overrightarrow{CA}| = 2|\overrightarrow{NC}|$, so that $\overrightarrow{NC} = -\frac{1}{2}\overrightarrow{CA}$ and $\lambda = -\frac{1}{2}$;

(b) $\overrightarrow{BD}$ and $\overrightarrow{BN}$ are in the same direction $|\overrightarrow{BD}| = 2|\overrightarrow{BN}|$, so that $\overrightarrow{BD} = 2\overrightarrow{BN}$ and $\lambda = 2$;

(c) $\overrightarrow{AC}$ and $\overrightarrow{CN}$ are in opposite directions, $|\overrightarrow{AC}| = 2|\overrightarrow{CN}|$, so that $\overrightarrow{AC} = -2\overrightarrow{CN}$ and $\lambda = -2$;

(d) since $|\overrightarrow{BB}| = 0$, and $|\overrightarrow{BD}| \neq 0$, we have $\overrightarrow{BB} = 0\overrightarrow{BD}$ and $\lambda = 0$. ▲

Example 2. Given two vectors **a** and **b** (Fig. 14), construct the vector $2\mathbf{a} - \frac{1}{2}\mathbf{b}$.

△ We construct the vectors $2\mathbf{a}$ and $-\frac{1}{2}\mathbf{b}$ and then add them together using the triangle rule. ▲

Theorem (a necessary and sufficient condition for collinearity). *In order for a vector* **a** *to be collinear with a nonzero vector* **b**, *it is neces-*

sary and sufficient that there exist a number λ *satisfying the condition*

$$\mathbf{a} = \lambda \mathbf{b}. \tag{4}$$

□ Let equality (4) be satisfied. Then the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear by the definition of the multiplication of a vector by a number and by the definition of collinearity. Thus, sufficiency has been proved.

When proving the necessity, we consider three cases: (a) the collinear vectors $\mathbf{a}$ and $\mathbf{b}$ are in the same direction, (b) in opposite directions, and (c) $\mathbf{a} = 0$. In the first case $\mathbf{a} = \frac{|\mathbf{a}|}{|\mathbf{b}|}\mathbf{b}$, and equality (4) occurs for $\lambda = \frac{|\mathbf{a}|}{|\mathbf{b}|}$. In the second case $\mathbf{a} = -\frac{|\mathbf{a}|}{|\mathbf{b}|}\mathbf{b}$, and equality (4) is satisfied for $\lambda = -\frac{|\mathbf{a}|}{|\mathbf{b}|}$. And if $\mathbf{a} = 0$, then $\mathbf{a} = 0 \cdot \mathbf{b}$, that is, equality (4) is fulfilled for $\lambda = 0$. Thus, the necessity has been proved. ■

Example 3. Show that the vectors $\overrightarrow{AB} + \overrightarrow{CB} + 2\overrightarrow{BA}$ and $5\overrightarrow{AC}$ are collinear.

△ Using the properties of the operations on vectors, we get the equality $\overrightarrow{AB} + \overrightarrow{CB} + 2\overrightarrow{BA} = (\overrightarrow{AB} + \overrightarrow{BA}) + (\overrightarrow{CB} + \overrightarrow{BA}) = 0 + \overrightarrow{CA} = \overrightarrow{CA}$. The given vector is collinear with the vector $5\overrightarrow{AC} = (-5)\overrightarrow{CA}$ by the proved theorem. ▲

A vector $\mathbf{a}$ is said to be a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, . . ., $\mathbf{a}_n$ if it is representable in the form

$$\mathbf{a} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_n \mathbf{a}_n,$$

where λ_1, λ_2, . . ., λ_n are some numbers.

Thus, the vector $\mathbf{a} = \mathbf{a}_1 + 2\mathbf{a}_2 - 3\mathbf{a}_3 + \frac{1}{7}\mathbf{a}_4$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

If a vector is represented in the form of a linear combination of some vectors, then we say that the vector is resolved into these vectors.

Example 4. Points M and N are midpoints of the sides of the parallelogram $ABCD$ (Fig. 15). Resolve the vector $\overrightarrow{DC}$ into the component vectors $\mathbf{a} = \overrightarrow{AM}$ and $\mathbf{b} = \overrightarrow{AN}$.

△ From the triangle AND we have:

$$\overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \mathbf{b}. \tag{5}$$

From the triangle ABM we find $\overrightarrow{AB}+\overrightarrow{BM}=\mathbf{a}$. Since $\overrightarrow{AB}=\overrightarrow{DC}$, $\overrightarrow{BM}=\frac{1}{2}\overrightarrow{BC}=\frac{1}{2}\overrightarrow{AD}$, we have

$$\overrightarrow{DC}+\frac{1}{2}\overrightarrow{AD}=\mathbf{a}. \tag{6}$$

Equality (5) implies that

$$\overrightarrow{AD}=\mathbf{b}-\frac{1}{2}\overrightarrow{DC}. \tag{7}$$

Substituting $\overrightarrow{AD}$ from (7) into (6), we get $\overrightarrow{DC}+\frac{1}{2}\mathbf{b}-\frac{1}{4}\overrightarrow{DC}=\mathbf{a}$, or $\frac{3}{4}\overrightarrow{DC}=\mathbf{a}-\frac{1}{2}\mathbf{b}$, where $\overrightarrow{DC}=\frac{4}{3}\mathbf{a}-\frac{2}{3}\mathbf{b}$. ▲

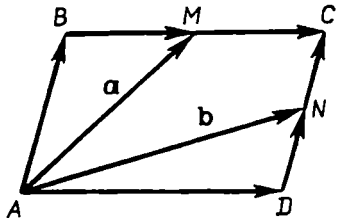

Fig. 15

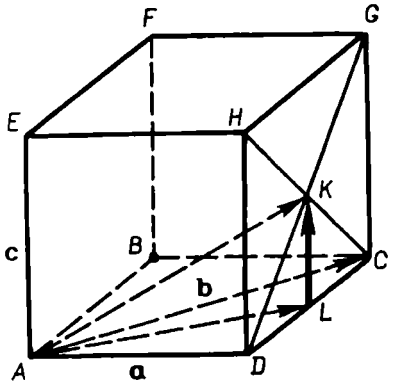

Fig. 16

Example 5. Given a cube $ABCDEFGH$ (Fig. 16). Resolve the vector $\overrightarrow{AK}$, where K is the centre of the face $DHGC$, into the vectors $\mathbf{a}=\overrightarrow{AD}$, $\mathbf{b}=\overrightarrow{AC}$, and $\mathbf{c}=\overrightarrow{AE}$.

△ Let L be the midpoint of the edge DC. Then $\overrightarrow{AL}=\overrightarrow{AD}+\overrightarrow{DL}=\mathbf{a}+\frac{1}{2}\overrightarrow{DC}$. From the triangle AKL we find that $\overrightarrow{AK}=\overrightarrow{AL}+\overrightarrow{LK}$. Since $\overrightarrow{DC}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{LK}=\frac{1}{2}\overrightarrow{AE}=\frac{1}{2}\mathbf{c}$, the desired resolution has the form: $\overrightarrow{AK}=\mathbf{a}+\frac{\mathbf{b}-\mathbf{a}}{2}+\frac{\mathbf{c}}{2}=\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}+\frac{1}{2}\mathbf{c}$. ▲

Exercises

1. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, construct the following vectors:
(a) $\mathbf{b}-\mathbf{a}$; (b) $-\mathbf{b}-\mathbf{a}$; (c) $\frac{1}{3}\mathbf{a}-3\mathbf{b}$; (d) $-\frac{1}{7}\mathbf{a}$.

2. A material point is acted upon by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Find the magnitude of their resultant if $|\mathbf{F}_1| = 5H$, $|\mathbf{F}_2| = 3H$, and the angle φ between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ is equal to 90°.

3. For any pentagon $ABCDE$ find the sum of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EA}$.

4. Given a tetrahedron $ABCS$. Find the sum of the vectors: (a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CS}$; (b) $\overrightarrow{AC} + \overrightarrow{CS} + \overrightarrow{SA} + \overrightarrow{AB}$.

5. In a rectangle $ABCD$ diagonals $\overrightarrow{DB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$ are drawn. Represent the vectors $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{BD}$, and $\overrightarrow{AD} + \overrightarrow{CD}$ as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$.

6. In a parallelogram $ABCD$ $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$, and O is the point of intersection of the diagonals. Resolve the vectors $\overrightarrow{BD}$, $\overrightarrow{OB}$, $\overrightarrow{AC}$, and $\overrightarrow{CD}$ into the vectors $\mathbf{a}$ and $\mathbf{b}$.

7. In an isosceles trapezoid $ABCD$ the size of the angle BAD is equal to 60°, $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CD}| = 2$, points M and N being the midpoints of the sides BC and DC. Decompose the vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{BC}$, $\overrightarrow{AM}$, $\overrightarrow{AN}$, and $\overrightarrow{MN}$ with respect to the unit vectors $\mathbf{e}_1 = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ and $\mathbf{e}_2 = \dfrac{\overrightarrow{AD}}{|AD|}$.

8. Given a cube $ABCDA_1B_1C_1D_1$. Find the resolutions of the vectors $\overrightarrow{AC_1}$, $\overrightarrow{AB_1}$, $\overrightarrow{D_1C_1}$, $\overrightarrow{B_1D_1}$, $\overrightarrow{BC}$, $\overrightarrow{C_1C}$, $\overrightarrow{B_1D}$, and $\overrightarrow{B_1O}$ (O the centre of the cube) into the vectors $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{AD}$, and $\mathbf{c} = \overrightarrow{AA_1}$.

1.2. Coordinates of a Vector and Coordinates of a Point

1. Rectangular Cartesian Coordinates in the Plane. To describe geometrical objects in a plane (such as points, line segments, lines, etc.), we introduce rectangular Cartesian coordinates in this plane by taking two mutually perpendicular number axes Ox and Oy having a common origin O. We say that these axes (Fig. 17) form a rectangular Cartesian coordinate system in the plane. The point O is called the origin of coordinates, the x- and y-axes are the coordinate axes (the x-axis being called the axis of abscissas, the y-axis, the axis of ordinates). As is customary, Figure 17 shows the axis of abscissas as the horizonatal axis and the axis of ordinates as the vertical axis. The unit vectors of the x- and y-axes are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively. The plane on which the coordinate system is specified is called the coordinate plane.

By definition, the vectors $\mathbf{i}$ and $\mathbf{j}$ are unit and orthogonal vectors: $|\mathbf{i}| = |\mathbf{j}| = 1$, $\mathbf{i} \perp \mathbf{j}$.

Definition 1. The projections of a vector on the coordinate axes are called the coordinates of the vector. The projection of the vector on the x-axis is termed the first coordinate or the x-coordinate, and its projection on the y-axis, the second coordinate or the y-coordinate.

Consequently, denoting by x_a, y_a the coordinates of the vector $\mathbf{a}$, according to the definition, we have

$$x_a = \mathrm{pr}_{Ox}\mathbf{a} = |\mathbf{a}| \cos \widehat{(\mathbf{a}, \mathbf{i})},$$

$$y_a = \mathrm{pr}_{Oy}\mathbf{a} = |\mathbf{a}| \cos \widehat{(\mathbf{a}, \mathbf{j})}.$$

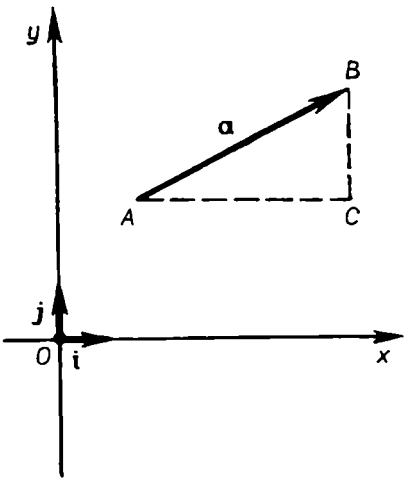

Fig. 17

Let a vector $\mathbf{a} = \overrightarrow{AB}$ be given in the plane. Through the point A, we draw a straight line parallel to the x-axis, and through the point B, a straight line parallel to the y-axis (Fig. 17). Let C be the point of intersection of the straight lines thus drawn. Then, obviously,

$$\overrightarrow{AC} = x_a\mathbf{i}, \quad \overrightarrow{CB} = y_a\mathbf{j} \text{ and}$$

$$\mathbf{a} = \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = x_a\mathbf{i} + y_a\mathbf{j}.$$

Hence we get the following assertion: if x_a, y_a are coordinates of the vector $\mathbf{a}$, then

$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}. \tag{1}$$

It is obvious that the converse is also true: if $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j}$, then x_a, y_a are coordinates of the vector $\mathbf{a}$ since

$$\mathrm{pr}_{Ox}(x_a\mathbf{i} + y_a\mathbf{j}) = x_a, \quad \mathrm{pr}_{Oy}(x_a\mathbf{i} + y_a\mathbf{j}) = y_a.$$

Vector equality (1) is often written in the symbolic form:

$$\mathbf{a} = \{x_a, y_a\}.$$

Equality (1) means that any vector in the plane can be resolved (or decomposed) into two mutually perpendicular vectors. A more general assertion states that any vector in the plane can be resolved into two noncollinear vectors. Bearing in mind the last fact, we can construct an oblique Cartesian coordinate system. However, we are not going to use this system.

2. Rectangular Cartesian Coordinates in Space. In space, a rectangular Cartesian coordinate system is formed by three mutually perpendicular axes with a common origin. One of the axes is called the

axis of abscissas and is denoted by Ox (x-axis), the second, the axis of ordinates and is denoted by Oy (y-axis), the third by Oz (z-axis). The unit vectors of the x-, y-, and z-axes are usually denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively.

The coordinates of a vector in space are defined as its projections on the coordinate axes. The projection of the vector on the x-axis is called the first or x-coordinate, the projection on the y-axis, the second or y-coordinate, and the projection on the z-axis, the third or z-coordinate.

Let in space a vector $\mathbf{a}$ be given with coordinates x_a, y_a, z_a. We can prove (much in the same way as it was done for the plane) that the following equality holds true:

$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}, \tag{1}$$

which is often written in the symbolic form

$$\mathbf{a} = \{x_a, y_a, z_a\}.$$

Equality (1) means that any vector in space can be resolved into three mutually perpendicular vectors. In the general case, any vector in space can be resolved into any three noncoplanar vectors on the basis of which we can construct an oblique Cartesian coordinate system.

3. Coordinates of the Sum and Difference of Two Vectors and the Product of a Vector by a Number. From the properties of the sum (difference) of two vectors and the product of a vector by a number it follows that if the vector $\mathbf{a}$ has the coordinates x_a, y_a, z_a, and the vector $\mathbf{b}$ the coordinates x_b, y_b, z_b, then

$$\begin{aligned}\mathbf{a} \pm \mathbf{b} &= x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k} \pm x_b\mathbf{i} \pm y_b\mathbf{j} \pm z_b\mathbf{k} \\ &= (x_a \pm x_b)\,\mathbf{i} + (y_a \pm y_b)\,\mathbf{j} + (z_a \pm z_b)\,\mathbf{k}\end{aligned}$$

and

$$\lambda\mathbf{a} = \lambda\,(x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) = \lambda x_a\mathbf{i} + \lambda y_a\mathbf{j} + \lambda z_a\mathbf{k}.$$

Thus, the vector $\mathbf{a} \pm \mathbf{b}$ has the coordinates $x_a \pm x_b$, $y_a \pm y_b$, $z_a \pm z_b$, that is,

$$\mathbf{a} \pm \mathbf{b} = \{x_a \pm x_b, \quad y_a \pm y_b, \quad z_a \pm z_b\}, \tag{1}$$

and the vector $\lambda\mathbf{a}$ has the coordinates λx_a, λy_a, λz_a, that is,

$$\lambda\mathbf{a} = \{\lambda x_a, \lambda y_a, \lambda z_a\}. \tag{2}$$

Consequently, as two vectors are added (subtracted), their coordinates are added (subtracted). As a vector is multiplied by a number, its coordinates are multiplied by this number.

Example. Given a vector $\mathbf{a} = \{1, -3, 5\}$. Find the unknown coordinates of the vector $\mathbf{b} = \{x, 6, z\}$ which is collinear with the vector $\mathbf{a}$.

△ According to the collinearity test (see 1.1), we have $\mathbf{b} = k\mathbf{a} = k(1\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$. Then, by formula (2), $\mathbf{b} = \{k, -3k, 5k\}$. Since $-3k = 6$, we have: $\mathbf{k} = -2$ and $\mathbf{b} = \{-2, 6, -10\}$. ▲

4. Coordinates of a Point. Let a Cartesian coordinate system $Oxyz$ be specified in space (Fig. 18). For any point A the coordinates of the vector $\overrightarrow{OA}$ are defined as the coordinates of the point A. The vector $\overrightarrow{OA}$ is called the radius vector of the point A. Consequently,

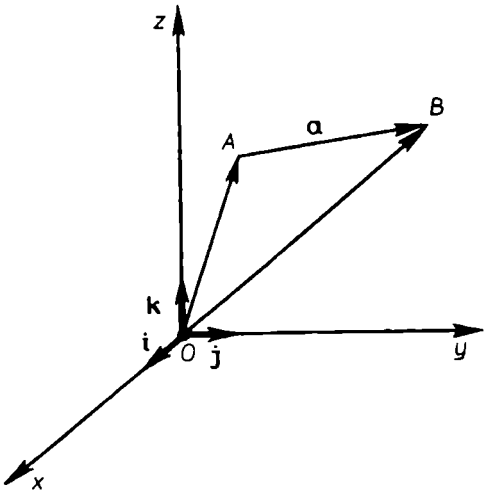

Fig. 18

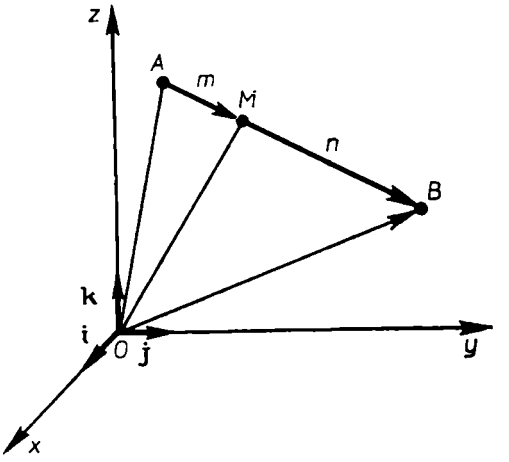

Fig. 19

the coordinates of a point are the coordinates of its radius vector. The coordinates of the point A are usually written in parentheses in the following way: $A(x_A, y_A, z_A)$.

Let us find the coordinates of the vector $\mathbf{a} = \overrightarrow{AB}$ if the coordinates of the points A and B are known. Let the point A have coordinates (x_A, y_A, z_A), and the point B coordinates (x_B, y_B, z_B). Then (see Fig. 18)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k} - x_A\mathbf{i} - y_A\mathbf{j} - z_A\mathbf{k}$$
$$= (x_B - x_A)\,\mathbf{i} + (y_B - y_A)\,\mathbf{j} + (z_B - z_A)\,\mathbf{k},$$

and therefore the vector $\overrightarrow{AB}$ has the following coordinates: $x_B - x_A$, $y_B - y_A$, $z_B - z_A$.

Consequently, the coordinates of a vector are equal to the differences between the appropriate coordinates of its terminus and origin:

$$\overrightarrow{AB} = \{x_B - x_A,\ y_B - y_A,\ z_B - z_A\}. \qquad (1)$$

Example 1. Find the coordinates of the vector $\overrightarrow{AB}$ if A (0, 1, 2) and B (3, 4, 21).

△ Let $\overrightarrow{AB} = \{x_{AB}, y_{AB}, z_{AB}\}$. Then, by formula (1), $x_{AB} = x_B - x_A = 3 - 0 = 3$; $y_{AB} = y_B - y_A = 4 - 1 = 3$; $z_{AB} = z_B - z_A = 21 - 2 = 19$. Thus, $\overrightarrow{AB} = \{3, 3, 19\}$. ▲

Example 2. Given two points M (1, −1, 3) and N (2, 0, 7). Find the projections of the vector $\overrightarrow{MN}$ on the coordinate axes.

△ According to formula (1), to find the coordinates of the vector $\overrightarrow{MN}$, we have to subtract the coordinates of its origin from the like coordinates of its terminus. As a result, we have $\overrightarrow{MN} = \{2 - 1, 0 - (-1), 7 - 3\} = \{1, 1, 4\}$. But the coordinates of a vector in a rectangular Cartesian coordinate system are just equal to the required projections of the vector on the coordinate axes. Thus, $\mathrm{pr}_{Ox}\overrightarrow{MN} = 1$, $\mathrm{pr}_{Oy}\overrightarrow{MN} = 1$, $\mathrm{pr}_{Oz}\overrightarrow{MN} = 4$. ▲

5. Coordinates of a Point Dividing a Line Segment in a Given Ratio. It is required to divide the given line segment AB (Fig. 19) in the given ratio $m:n$, that is, to find the coordinates of a point M (x_M, y_M, z_M) on the line segment AB such that

$$\frac{|\overrightarrow{AM}|}{|\overrightarrow{MB}|} = \frac{m}{n}.$$

It is obvious that the point M divides the line segment AB in the given ratio $m:n$ if and only if

$$\overrightarrow{AM} = \frac{m}{n}\overrightarrow{MB}. \tag{1}$$

We express the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ in terms of radius vectors $\overrightarrow{OA}$, $\overrightarrow{OM}$, and $\overrightarrow{OB}$. Then equation (1) takes the form $\overrightarrow{OM} - \overrightarrow{OA} = \frac{m}{n}(\overrightarrow{OB} - \overrightarrow{OM})$. Hence

$$\overrightarrow{OM} = \frac{n}{m+n}\overrightarrow{OA} + \frac{m}{m+n}\overrightarrow{OB}. \tag{2}$$

Formula (2) yields the solution of the problem since it expresses the radius vector of the desired point M, dividing the line segment AB in the given ratio $m:n$, in terms of the radius vectors of the given points A (x_A, y_A, z_A) and B (x_B, y_B, z_B). Vector equality (2) is

equivalent to three scalar equalities

$$x_M = \frac{n}{m+n} x_A + \frac{m}{m+n} x_B, \quad y_M = \frac{n}{m+n} y_A + \frac{m}{m+n} y_B,$$

$$z_M = \frac{n}{m+n} z_A + \frac{m}{m+n} z_B. \tag{3}$$

In the particular case when M is the midpoint of the segment AB, formula (3) takes the form:

$$x_M = \frac{x_A + x_B}{2}, \quad y_M = \frac{y_A + y_B}{2}, \quad z_M = \frac{z_A + z_B}{2}. \tag{4}$$

Example 1. Prove that the medians of an arbitrary triangle ABC intersect at a point M such that: (a) the point M divides each median in the ratio 2:1 and is two-thirds of the distance from a vertex to the opposite side along a median; (b) for any point O (in the plane) the following relation is valid:

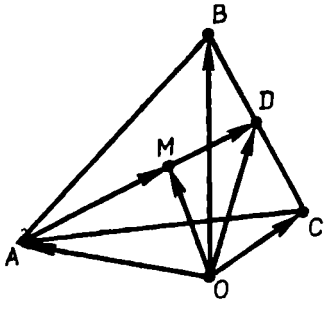

Fig. 20

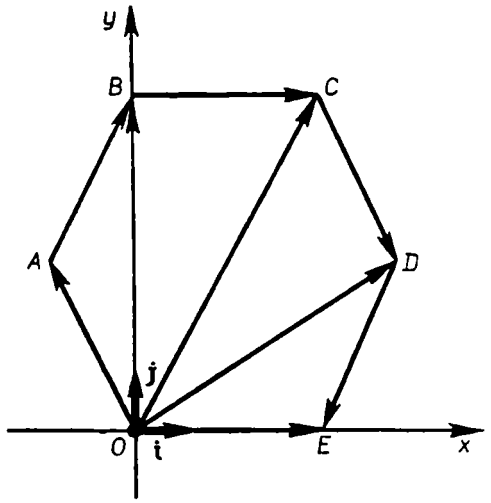

Fig. 21

$$\overrightarrow{OM} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

△ On the median AD, consider the point M which divides it in the ratio 2:1 as counted from the vertex of the triangle (Fig. 20). Then, by formula (2), we get $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}$, where O is an arbitrary point of the plane. The point D is the midpoint of the side BC, therefore, according to formula (2) $(m = n)$, $\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}$. Consequently, $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3} \times \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. The same result can also be

obtained for any other median of the triangle ABC. This means that the point M is common to all the three medians. Thereby both assertions have been proved. ▲

Exercises

1. Given in a rectangular Cartesian coordinate system are two points $A(3, -4, 5)$ and $B(-1, 2, -2)$. Find the coordinates of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$.
2. Determine α and β for which the vectors $\mathbf{a} = \{-2, 3, \beta\}$ and $\mathbf{b} = \{\alpha, -6, 2\}$ are collinear.
3. Are the points $A(3, -1, 2)$, $B(1, 2, -1)$, $C(-1, 1, -3)$, and $D(3, -5, 3)$ the vertices of a trapezoid?
4. A regular hexagon is given in the plane. Resolve all the vectors represented in Fig. 21 into component vectors with respect to the unit vectors $\mathbf{i}$ and $\mathbf{j}$ if $|\overrightarrow{OE}| = 4$.
5. The point $C(2, 3)$ divides the line segment AB in the ratio 1:2. Find the coordinates of the point B if the coordinates of the point A are known: $x = 1$, $y = 2$.

1.3. Scalar Product (Dot Product, Inner Product) of Two Vectors

1. Definition of Scalar Product of Two Vectors. In physics, when considering a rectilinear displacement $\overrightarrow{BC}$ (from the initial point B to the terminal point C), the work A done by a constant force $\mathbf{F}$ is computed by the formula

$$A = |\mathbf{F}|\,|\overrightarrow{BC}|\cos\varphi,$$

where φ is the angle between the vector of the force $\mathbf{F}$ and displacement $\overrightarrow{BC}$. This formula associates a scalar quantity (number), that is, work with the vectors $\mathbf{F}$ and $\overrightarrow{BC}$. The quantity A is called the scalar product of the vectors $\mathbf{F}$ and $\overrightarrow{BC}$. A scalar product can be defined for any two vectors. It is widely used in physics and mathematics.

Definition. The scalar product of two nonzero vectors is a number equal to the product of the lengths of these vectors by the cosine of the angle between them. If at least one of the vectors is a null vector, then the scalar product of these vectors is regarded to be equal to zero.

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a}\cdot\mathbf{b}$ (alternative notation $(\mathbf{ab})$). Thus, by definition

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\varphi, \tag{1}$$

where φ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a} = \mathbf{b}$, then the scalar product takes the form $\mathbf{a} \cdot \mathbf{a}$ and is called the scalar square of the vector $\mathbf{a}$ (written: $\mathbf{a}^2$). It is obvious that

$$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2.$$

Example. It is known that $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, and the angle φ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to 135°. Find $\mathbf{a} \cdot \mathbf{b}$.

△ By formula (1), we find:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \varphi = 1 \cdot 2 \cdot \cos 135° = 2 \cos (90° + 45°)$$

$$= -2 \sin 45° = -2 \frac{\sqrt{2}}{2} = -\sqrt{2}. \blacktriangle$$

2. Properties of Scalar Product of Vectors.

1. For any two vectors $\mathbf{a}$ and $\mathbf{b}$ the following equality holds:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \tag{1}$$

This is the commutative property of the scalar product.

2. For any two vectors $\mathbf{a}$, $\mathbf{b}$ and any number k the following equality holds true:

$$(k\mathbf{a}) \cdot \mathbf{b} = k (\mathbf{a} \cdot \mathbf{b}). \tag{2}$$

Consequently, a numerical factor may be taken outside the sign of the scalar product.

3. For any three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the following equality is valid:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \tag{3}$$

This is the distributive property of the scalar product.

□ The first two properties almost directly follow from the definition of the scalar product. Indeed, by definition, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \varphi$, and $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}||\mathbf{a}| \cos \varphi = |\mathbf{a}||\mathbf{b}| \cos \varphi = \mathbf{a} \cdot \mathbf{b}$. Thus, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, and equality (1) has been proved.

Let us now prove equality (2). If $k > 0$, then the angle φ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to the angle ψ between the vectors $k\mathbf{a}$ and $\mathbf{b}$ so that $(k\mathbf{a}) \cdot \mathbf{b} = |k\mathbf{a}||\mathbf{b}| \cos \psi = k |\mathbf{a}||\mathbf{b}| \cos \varphi = k \cdot (\mathbf{a} \cdot \mathbf{b})$. If $k < 0$, then the angle ψ between the vectors $k\mathbf{a}$ and $\mathbf{b}$ is equal to $(180° - \varphi)$, where φ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Then $(k\mathbf{a}) \cdot \mathbf{b} = |k\mathbf{a}||\mathbf{b}| \cos \psi = |k||\mathbf{a}||\mathbf{b}| \cos (180° - \varphi) = -k |\mathbf{a}||\mathbf{b}| (-\cos \varphi) = k |\mathbf{a}||\mathbf{b}| \cos \varphi = k (\mathbf{a} \cdot \mathbf{b})$. If $k = 0$ or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $(k\mathbf{a}) \cdot \mathbf{b} = 0$ and $k (\mathbf{a} \cdot \mathbf{b}) = 0$. Consequently, in this case also $(k\mathbf{a}) \cdot \mathbf{b} = k (\mathbf{a} \cdot \mathbf{b})$.

To prove property 3, let us note that if $\mathbf{a} = \mathbf{0}$, then equality (3) is obvious; and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}| \operatorname{pr}_l (\mathbf{b} + \mathbf{c})$, where l is an axis with unit vector $\mathbf{e} = \frac{\mathbf{a}}{|\mathbf{a}|}$. Since for any two vectors $\mathbf{b}$ and $\mathbf{c}$ the equality $\operatorname{pr}_l (\mathbf{b} + \mathbf{c}) = \operatorname{pr}_l \mathbf{b} + \operatorname{pr}_l \mathbf{c}$ is valid (see 1.1), we have $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}| (\operatorname{pr}_l \mathbf{b} + \operatorname{pr}_l \mathbf{c}) = |\mathbf{a}| \operatorname{pr}_l \mathbf{b} + |\mathbf{a}| \operatorname{pr}_l \mathbf{c} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, and equality (3) has been proved. ■

Note that formulas (1) and (3) imply the formula

$$(\mathbf{a} + \mathbf{b})\cdot\mathbf{c} = \mathbf{ac} + \mathbf{bc}. \tag{4}$$

The similarity of the properties of a scalar product of vectors to those of a product of real numbers enables us to easily perform various computations and transformations of scalar products.

Example 1. Prove the equality $(\mathbf{a} \pm \mathbf{b})^2 = \mathbf{a}^2 \pm 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2$.

△ Using the properties of the scalar product, we get

$$\begin{aligned}(\mathbf{a} \pm \mathbf{b})^2 &= (\mathbf{a} \pm \mathbf{b})\cdot(\mathbf{a} \pm \mathbf{b}) = (\mathbf{a} \pm \mathbf{b})\cdot\mathbf{a} \pm (\mathbf{a} \pm \mathbf{b})\cdot\mathbf{b} \\ &= \mathbf{a}\cdot\mathbf{a} \pm \mathbf{b}\cdot\mathbf{a} \pm \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = \mathbf{a}^2 \pm \mathbf{a}\cdot\mathbf{b} \pm \mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2 \\ &= \mathbf{a}^2 \pm 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2. \blacktriangle\end{aligned}$$

Theorem. *Two nonzero vectors are mutually perpendicular if and only if their scalar product is equal to zero.*

□ Indeed, if $\mathbf{a} \perp \mathbf{b}$, then $\cos(\widehat{\mathbf{a}, \mathbf{b}}) = 0$, and, consequently, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos 90^\circ = 0$.

And vice verse, if $\mathbf{a}\cdot\mathbf{b} = 0$, then from the equality $|\mathbf{a}|\,|\mathbf{b}| \times \cos(\widehat{\mathbf{a}, \mathbf{b}}) = 0$ it follows that $\cos(\widehat{\mathbf{a}, \mathbf{b}}) = 0$, since $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$. Consequently, $(\widehat{\mathbf{a}, \mathbf{b}}) = 90^\circ$, that is $\mathbf{a} \perp \mathbf{b}$. ■

Example 2. What condition must be satisfied by two unequal vectors $\mathbf{a}$ and $\mathbf{b}$ so that the vector $\mathbf{a} + \mathbf{b}$ is perpendicular to the vector $\mathbf{a} - \mathbf{b}$?

△ The vector $\mathbf{a} + \mathbf{b}$ will be perpendicular to the vector $\mathbf{a} - \mathbf{b}$ if and only if $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) = 0$. We take advantage of the properties of the scalar product:

$$\begin{aligned}(\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} - \mathbf{b}) &= (\mathbf{a} + \mathbf{b})\cdot\mathbf{a} - (\mathbf{a} + \mathbf{b})\cdot\mathbf{b} = \\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{b} \\ &= |\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 = |\mathbf{a}|^2 - |\mathbf{b}|^2,\end{aligned}$$

whence it follows that the vector $\mathbf{a} + \mathbf{b}$ is perpendicular to the vector $\mathbf{a} - \mathbf{b}$ if $|\mathbf{a}| = |\mathbf{b}|$. ▲

3. Scalar Product of Vectors Represented by Their Coordinates. Let there be specified a rectangular Cartesian coordinate system with unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and let there be given two vectors $\mathbf{a} = \{x_a, y_a, z_a\}$ and $\mathbf{b} = \{x_b, y_b, z_b\}$. Since $\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$ and $\mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}$, and bearing in mind the properties of scalar multiplication of vectors, we get

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k})\cdot(x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) \\ &= x_a x_b\mathbf{i}^2 + y_a y_b\mathbf{j}^2 + z_a z_b\mathbf{k}^2 + x_a y_b\mathbf{ij} + x_a z_b\mathbf{ik} + y_a x_b\mathbf{ji} \\ &\quad + y_a z_b\mathbf{jk} + z_a x_b\mathbf{ki} + z_a y_b\mathbf{kj}.\end{aligned}$$

Since $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$ and $\mathbf{ij} = \mathbf{ji} = \mathbf{ik} = \mathbf{ki} = \mathbf{jk} = \mathbf{kj} = 0$, we found that

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b. \tag{1}$$

In words, the scalar product of two vectors is equal to the sum of the products of the like coordinates of these vectors. Obviously, for two vectors $\mathbf{a} = \{x_a, y_a\}$ and $\mathbf{b} = \{x_b, y_b\}$ given in the plane we have the formula

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b. \tag{2}$$

Example 1. Compute $\mathbf{a} \cdot \mathbf{b}$ if $\mathbf{a} = \mathbf{i} + 4\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$.

△ By the properties of the scalar product, $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} + 4\mathbf{j})(-2\mathbf{i} + \mathbf{j}) = -2\mathbf{i}^2 + 4\mathbf{j}^2 = -2 + 4 = 2$. ▲

Example 2. Compute $\mathbf{a} \cdot \mathbf{b}$ if $\mathbf{a} = \{1, 2, 3\}$ and $\mathbf{b} = \{-3, -2, -1\}$.

△ By formula (1), we have $\mathbf{a} \cdot \mathbf{b} = 1(-3) + 2(-2) + 3(-1) = -10$. ▲

Setting $\mathbf{b} = \mathbf{a}$ in formula (1), we get the relation $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = x_a^2 + y_a^2 + z_a^2$. Hence

$$|\mathbf{a}| = \sqrt{x_a^2 + y_a^2 + z_a^2}, \quad \text{where} \quad \mathbf{a} = \{x_a, y_a, z_a\}, \tag{3}$$

that is, the absolute value (length) of a vector is equal to the square root of the sum of squares of its coordinates. For a vector in the plane from formula (2) we find a similar formula:

$$|\mathbf{a}| = \sqrt{x_a^2 + y_a^2}, \quad \text{where} \quad \mathbf{a} = \{x_a, y_a\}. \tag{4}$$

Example 3. Determine the absolute values of the sum and difference of the vectors $\mathbf{a} = \{3, -5, 8\}$ and $\mathbf{b} = \{-1, 1, -4\}$.

△ We compute the coordinates of the sum and difference of the vectors:

$$\mathbf{a} + \mathbf{b} = \{3 - 1, \quad -5 + 1, \quad 8 - 4\} = \{2, -4, 4\},$$
$$\mathbf{a} - \mathbf{b} = \{3 + 1, \quad -5 - 1, \quad 8 + 4\} = \{4, -6, 12\}.$$

Further, we make use of formula (3):

$$|\mathbf{a} + \mathbf{b}| = \sqrt{2^2 + (-4)^2 + 4^2} = \sqrt{36} = 6,$$
$$|\mathbf{a} - \mathbf{b}| = \sqrt{4^2 + (-6)^2 + 12^2} = \sqrt{196} = 14. \blacktriangle$$

Since the coordinates of the vector $\mathbf{a} = \overrightarrow{AB}$ are equal to the differences of the corresponding coordinates of its origin and terminus, the distance d between the points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ equal to the length of the vector $\mathbf{a} = \{x_B - x_A, y_B - y_A, z_B - z_A\}$ is computed by the formula

$$d = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}.$$

Example 4. On the x-axis, find the point B whose distance from the point A (3, 4) is equal to 5.

△ Since the distance from the point $A\ (3, 4)$ to the point $B\ (x_B, 0)$ located on the x-axis is equal to 5, the equality $5 = \sqrt{(x_B - 3)^2 + (0 - 4)^2}$ is valid, whence we find that either $x_B = 6$ or $x_B = 0$. Thus, there are two points on the x-axis, $B_1\ (6, 0)$ and $B_2\ (0, 0)$, whose distance to the point $A\ (3, 4)$ is 5. ▲

The scalar product of vectors is widely used in geometrical problems for computing angles and projections.

Example 5. Given two vectors $\mathbf{a} = \{3, 4\}$ and $\mathbf{b} = \{1, 2\}$. Find the angle φ between them.

△ By the definition of the scalar product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \varphi.$$

Making use of formula (4), we compute the absolute values of the vectors $\mathbf{a}$ and $\mathbf{b}$: $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$, and $|\mathbf{b}| = \sqrt{1^2 + 2^2} = \sqrt{5}$. The scalar product of the vectors whose coordinates are known can be computed by formula (2): $\mathbf{a} \cdot \mathbf{b} = 3 \cdot 1 + 4 \cdot 2 = 11$. We substitute the found values into the formula of scalar product: $11 = 5\sqrt{5} \cos \varphi$, whence $\cos \varphi = \dfrac{11}{5\sqrt{5}}$ and $\varphi = \arccos\left(\dfrac{11}{5\sqrt{5}}\right)$. ▲

Example 6. Compute the obtuse angle φ formed by the medians drawn from the acute angles of a right isosceles triangle (Fig. 22).

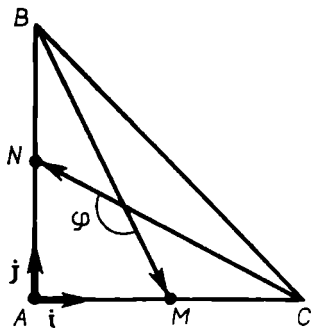

Fig. 22

△ We introduce two mutually perpendicular unit vectors $\mathbf{i}$ and $\mathbf{j}$, as is shown in Fig. 22. The desired angle φ is equal to the angle between the vectors $\overrightarrow{BM}$ and $\overrightarrow{CN}$. It enters the expression for the scalar product of the vectors $\overrightarrow{BM}$ and $\overrightarrow{CN}$: $\overrightarrow{BM} \cdot \overrightarrow{CN} = |\overrightarrow{BM}|\,|\overrightarrow{CN}| \cos \varphi$, whence

$$\cos \varphi = \frac{\overrightarrow{BM} \cdot \overrightarrow{CN}}{|\overrightarrow{BM}|\,|\overrightarrow{CN}|}. \qquad (5)$$

We have $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} = -k\mathbf{j} + \frac{k}{2}\mathbf{i}$, where k denotes the length of equal legs. Similarly $\overrightarrow{CN} = \overrightarrow{CA} + \overrightarrow{AN} = -k\mathbf{i} + \frac{k}{2}\mathbf{j}$. Then $\overrightarrow{BM} \times \overrightarrow{CN} = \left(-k\mathbf{j} + \frac{k}{2}\mathbf{i}\right)\left(-k\mathbf{i} + \frac{k}{2}\mathbf{j}\right) = -\frac{k^2}{2} - \frac{k^2}{2} = -k^2$; $|\overrightarrow{BM}| = \sqrt{(-k)^2 + \left(\frac{k}{2}\right)^2} = \frac{\sqrt{5}}{2}k = |\overrightarrow{CN}|$. Finally, from formula (5) we find:

$$\cos \varphi = \frac{-k^2}{\left(\frac{\sqrt{5}}{2}k\right)\left(\frac{\sqrt{5}}{2}k\right)} = -\frac{4}{5} \quad \text{and} \quad \varphi = \arccos\left(-\frac{4}{5}\right). \text{ ▲}$$

Example 7. Given the vertices of a triangle: A (1, 2), B (3, 4), and C (6, 2). Determine its interior angle at the vertex A.

△ The interior angle at the vertex A is equal to the angle φ between the vectors $\overrightarrow{AB} = \{3 - 1, 4 - 2\} = \{2, 2\}$ and $\overrightarrow{AC} = \{6 - 1, 2 - 2\} = \{5, 0\}$. Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}| \cos\varphi$, we have

$$\cos\varphi = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{2 \cdot 5 + 2 \cdot 0}{\sqrt{2^2 + 2^2} \cdot \sqrt{5^2 + 0^2}} = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \varphi = 45°. \ ▲$$

Exercises

1. Given two vectors $\mathbf{a} = \{2, -1, 0\}$ and $\mathbf{b} = \{2, 3, 1\}$. Compute: (a) $\mathbf{a} \cdot \mathbf{b}$; (b) $\mathbf{b}^2$; (c) $(\mathbf{a} + \mathbf{b})^2$; (d) $(2\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + 2\mathbf{b})$.

2. Given a vector $\mathbf{a} = \{2, -3\}$. Find the coordinates of the unit vectors perpendicular to the vector $\mathbf{a}$.

3. Given a vector $\mathbf{a} = \{3, -5\}$. Find the coordinates of some vector perpendicular to the vector $\mathbf{a}$. How many solutions does the problem have?

4. Find the values of α for which the following vectors are mutually perpendicular: (a) $\mathbf{a} = \alpha\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \alpha\mathbf{j} - 7\mathbf{k}$; (b) $\mathbf{a} = \{\alpha, -3, 2\}$ and $\mathbf{b} = \{1, 2, -\alpha\}$.

5. Find the values of α and β for which the vectors $\mathbf{a} = \{3, -1, \alpha\}$ and $\mathbf{b} = \{2, \beta, 1\}$ are mutually perpendicular if $|\mathbf{b}| = 3$.

6. In the xy-plane, find the vector $\mathbf{b}$ perpendicular to the vector $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ if $|\mathbf{b}| = |\mathbf{a}|$.

7. Find the vector $\mathbf{b}$ collinear with the vector $\mathbf{a}$ and satisfying the given condition: (a) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} \cdot \mathbf{a} = 3$; (b) $\mathbf{a} = \{-1, 2, 2\}$, $\mathbf{b} \cdot \mathbf{a} = -2$.

8. Given three vectors: $\mathbf{a} = \{2, -1, 3\}$, $\mathbf{b} = \{1, -3, 2\}$, and $\mathbf{c} = \{3, 2, -4\}$. Find the vector $\mathbf{d}$ satisfying the conditions $\mathbf{d} \cdot \mathbf{a} = -5$, $\mathbf{d} \cdot \mathbf{b} = -11$, and $\mathbf{d} \cdot \mathbf{c} = 20$.

9. Find the cosine of the angle between the vector $\mathbf{a} = \{3, -4\}$ and the x-axis.

10. Find the cosines of the angles between the vector $\mathbf{a} = \{3, -4, 12\}$ and each of the coordinate axes.

11. Find the angle between the diagonals of the parallelogram constructed on the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = -\mathbf{j} + 2\mathbf{k}$.

12. Find the angles of the triangle whose vertices are: A (5, 0, 0), B (1, 1, 1), and C (3, −1, 2).

13. Given three successive vertices of a parallelogram: A (−3, −2, 0), B (3, −3, 1), and C (5, 0, 2). Find the fourth vertex D and the angle between the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

14. Given a triangle with vertices A (3, −2, 1), B (3, 0, 2), and C (1, 2, 5). Compute the angle between the median BD and the side AC.

15. Given a quadrilateral with vertices at the points A (2, −1, 3), B (1, 4, 0), C (−4, 1, 1), and D (−5, −5, 3). Find the angle between the diagonals AC and BD.

16. Given a triangle with vertices at the points A (−1, 4, 1), B (3, 4, −2), and C (5, 2, −1). Compute the cosine of the angle at the vertex B.

17. Given a triangle with vertices at the points A (−3, 6), B (9, −10) and C (−5, 4). Determine the coordinates of the centre of the circle circumscribed about the triangle and its radius.

1.4. Second- and Third-order Determinants

1. Second-order Determinants. Consider an array of type

$$\begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}, \tag{1}$$

where a_1, b_1, a_2, and b_2 are some numbers. Any of such arrays is called the matrix of the second order, the numbers a_1, b_1, a_2, and b_2 being termed the elements of the matrix.

The number equal to $a_1 b_2 - a_2 b_1$ is called the determinant of matrix (1) or the determinant of the second order and is denoted by

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \text{ or } \det \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}.$$

Thus, by definition

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1. \tag{2}$$

Example 1. Evaluate the determinant of the matrix $\begin{pmatrix} 2 & -1 \\ -4 & -3 \end{pmatrix}$.

△ According to formula (2), we have:

$$\det \begin{pmatrix} 2 & -1 \\ -4 & -3 \end{pmatrix} = 2 \cdot (-3) - (-4) \cdot (-1) = -10. \ ▲$$

Example 2. Evaluate the determinant $\begin{vmatrix} \cos\alpha & (-\sin\alpha) \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

△ We compute by formula (2):

$$\begin{vmatrix} \cos\alpha & (-\sin\alpha) \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha - (-\sin^2\alpha) = \cos^2\alpha + \sin^2\alpha = 1. \ ▲$$

Example 3. Solve the equation $\begin{vmatrix} 2(x-4) \\ 1 \quad 3 \end{vmatrix} = 0$.

△ Since $\begin{vmatrix} 2(x-4) \\ 1 \quad 3 \end{vmatrix} = 2 \cdot 3 - 1 \cdot (x-4) = 10 - x$, the equation takes the form $10 - x = 0$, whence $x = 10$. ▲

2. Third-order Determinants. Consider a square array of type

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \tag{1}$$

where a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, and c_3 are some numbers. Such an array is called a matrix of the third order.

The determinant of matrix (1) or the determinant of the third order is denoted as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and is expressed in terms of second-order determinants in the following way:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \tag{2}$$

Expanding the second-order determinants by formula (2) from the preceding subsection, we find that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1 a_2 b_3 - c_1 a_3 b_2. \tag{3}$$

It is easier to memorize formula (2), as compared with formula (3), if we note the following rule for forming terms on the right-hand side of equality (2): we take the first element of the first row of matrix (1), that is, a_1, and multiply it by the determinant of the second-order matrix obtained from the original matrix given by (1) by striking out the row and column in which the taken element stands; we then take with the minus sign the second element of the first row, that is, b_1 and multiply it by the determinant of the second-order matrix obtained by deleting from the original matrix (1) the second column and first row (their intersection yields the element b_1), we then take the third element of the first row, that is, c_1 and multiply it by the corresponding second-order determinant.

The described rule and formula (2) are called the expansion of a third-order determinant in the elements of the first row.

Example 1. Evaluate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

△ We make use of formula (2) for third-order determinants:

$$\det A = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45 - 48) - 2 \cdot (36 - 42) + 3 \cdot (32 - 35) = -3 + 12 - 9 = 0. \ \blacktriangle$$

Third-order determinants possess the following properties:

1. The magnitude of a determinant remains unchanged if each of the rows is substituted by a column of the same position number, that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{4}$$

2. If any two rows (or columns) of a determinant are interchanged, its absolute value remains unaltered, while the sign is reversed, as if the determinant were multiplied by the number (—1).

3. A determinant with two identical rows (columns) is equal to zero.

4. If all the elements of a row (a column) are multiplied by a number k, then the determinant becomes multiplied by this number k.

5. If all the elements of a row (a column) are equal to zero, then the determinant itself is zero.

6. If the elements of one row (column) are respectively proportional to the elements of the other row (column) then the determinant is equal to zero.

□ Formula (4) is proved if the determinants on both sides of (4) are expanded by formula (3) or (2). Property 2 is proved in a similar way. To prove property 3, let us note that if two identical rows (columns) are interchanged, then, on the one hand, the determinant remains unchanged and, on the other hand, by virtue of property 2, its sign reverses. Consequently, it is equal to zero. To prove property 4, it suffices to note that the determinant is expressed in the form of sum (3) each term of which contains one and only one element from each row and one and only one element from each column. Thus, the number k will be a common factor of these terms, and therefore it may be factored out. Property 5 follows from property 4 (for $k = 0$). To prove property 6, let us note that, by virtue of property 4, the proportionality factor may be taken outside the sign of the

determinant. This done, we get a determinant with two identical rows, which is, by property 3, equal to zero. ■

Remark. Property 1 establishes complete equivalence of rows and columns. Therefore it is only natural to formulate the rest of the properties and their proofs either for rows or for columns.

Using properties 1-6, we can simplify the computation of third-order determinants.

Example 2. Compute the following determinants:

$$\Delta_1=\begin{vmatrix} 18 & 9 & 27 \\ 6 & 12 & 12 \\ 13 & 26 & 39 \end{vmatrix}, \quad \Delta_2=\begin{vmatrix} 43 & 86 & 43 \\ -1 & 8 & 2 \\ 1 & 2 & 1 \end{vmatrix}, \quad \Delta_3=\begin{vmatrix} 1 & 297 & 313 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}.$$

△ Taking the common factors of the first, second, and third rows outside the sign of the first determinant, we avoid the necessity to multiply and add together large numbers

$$\Delta_1=9\cdot 6\cdot 13\cdot\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}=702\left(2\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix}-\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}+3\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}\right)$$

$$=702\,(2\cdot 2-1+3\cdot 0)=702\cdot 3=2106.$$

There is no need to evaluate the second determinant by formula (2), since we see that its first and third rows are proportional to each other (the proportionality factor being equal to 43), and this determinant is equal to zero. Thus, $\Delta_2 = 0$. To evaluate the third determinant, we take advantage of property 1:

$$\Delta_3=\begin{vmatrix} 1 & 297 & 313 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ 297 & 1 & 2 \\ 313 & 2 & 1 \end{vmatrix}.$$

It is convenient to expand the last determinant in elements of the first row (formula (2)), since two of its three elements are equal to zero:

$$\Delta_3=1\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}+0+0=-3. \ ▲$$

3. Solving Systems of Linear Equations with the Aid of Determinants. Consider the system of three linear equations in three unknowns:

$$\begin{cases} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2, \\ a_3x+b_3y+c_3z=d_3. \end{cases} \tag{1}$$

The coefficients of the left-hand sides of the equations of system (1) form the matrix

$$A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{2}$$

This is the matrix of the system. The numbers on the right-hand sides of the equations form a column which is called the column of absolute terms. Recall that a solution of system (1) is defined as a collection of three numbers α, β, γ such that each equation of the system turns into an identity after substituting into it the numbers α, β, γ for the respective unknowns x, y, z.

In the general case, a system can have one solution, infinitely many solutions, and no solution at all.

For example, the system

$$\begin{cases} x+y+z=0, \\ 2x+2y+2z=2, \\ 3x+3y+3z=9 \end{cases} \tag{3}$$

has no solution, since if a solution existed, then $x+y+z$ would be simultaneously equal to zero, unity, and three.

Systems having no solution are said to be incompatible, otherwise they are called compatible.

The following theorem holds true:

Theorem. *System of equations* (1) *has a unique solution if and only if the determinant of its matrix is different from zero.*

In this case, the solution is found by Cramer's rule:

$$x=\frac{\det A_1}{\det A}, \quad y=\frac{\det A_2}{\det A}, \quad z=\frac{\det A_3}{\det A}, \tag{4}$$

where the matrices A_1, A_2 and A_3 are equal to

$$A_1=\begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}, \quad A_2=\begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}, \quad A_3=\begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix},$$

that is, these matrices are obtained from the matrix of system A by substituting the column of absolute terms for the first, second, and third column, respectively.

Example 1. Solve the system of equations

$$\begin{cases} 2x+y=5, \\ x+3z=16, \\ 5y-z=10. \end{cases}$$

△ We write out matrices A, A_1, A_2, and A_3 for the given system of equations:

$$A=\begin{pmatrix}2 & 1 & 0\\ 1 & 0 & 3\\ 0 & 5 & -1\end{pmatrix},\quad A_1=\begin{pmatrix}5 & 1 & 0\\ 16 & 0 & 3\\ 10 & 5 & -1\end{pmatrix},\quad A_2=\begin{pmatrix}2 & 5 & 0\\ 1 & 16 & 3\\ 0 & 10 & -1\end{pmatrix},$$

$$A_3=\begin{pmatrix}2 & 1 & 5\\ 1 & 0 & 16\\ 0 & 5 & 10\end{pmatrix}.$$

We evaluate their determinants:

$$\det A=2\begin{vmatrix}0 & 3\\ 5 & -1\end{vmatrix}-\begin{vmatrix}1 & 3\\ 0 & -1\end{vmatrix}=-30+1=-29,$$

$$\det A_1=5\begin{vmatrix}0 & 3\\ 5 & -1\end{vmatrix}-\begin{vmatrix}16 & 3\\ 10 & -1\end{vmatrix}=-75+46=-29,$$

$$\det A_2=2\begin{vmatrix}16 & 3\\ 10 & -1\end{vmatrix}-5\begin{vmatrix}1 & 3\\ 0 & -1\end{vmatrix}=-92+5=-87,$$

$$\det A_3=\det\begin{pmatrix}2 & 1 & 5\\ 1 & 0 & 16\\ 0 & 5 & 10\end{pmatrix}=\det\begin{pmatrix}2 & 1 & 0\\ 1 & 0 & 5\\ 5 & 16 & 10\end{pmatrix}$$

$$=2\begin{vmatrix}0 & 5\\ 16 & 10\end{vmatrix}-\begin{vmatrix}1 & 5\\ 5 & 10\end{vmatrix}=-160+15=-145.$$

Then, by formulas (4), we have:

$$x=\frac{-29}{-29}=1,\quad y=\frac{-87}{-29}=3,\quad z=\frac{-145}{-29}=5.\ \blacktriangle$$

In cases when $\det A=0$, system of equations (1) can have either no solution (be incompatible), as in the case of system (3) in which

$$\det A=\begin{vmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{vmatrix}=0,$$

and the equations contradict one another, or infinitely many solutions when individual equations of the system do not contradict one another. For example, the system of equations

$$\begin{cases}x+y+z=1,\\ 2x+2y+2z=2,\\ 3x+3y+3z=3,\end{cases}$$

in which $\det A = 0$ has an infinite set of solutions $x = u$, $y = v$, $z = 1 - u - v$, where u and v are arbitrary numbers.

Now, consider a homogeneous system of three linear equations in three unknowns

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0. \end{cases}$$

It is obvious that this system always has a so-called trivial solution $x=0$, $y=0$, and $z=0$. If in this system $\det A \neq 0$, then, by virtue of the above theorem, the trivial solution is unique. A homogeneous system of linear equations can have a nonzero (nontrivial) solution only in the case when $\det A = 0$. In this case, it turns out that it always has infinitely many solutions.

Example 2. Find all the solutions of the system

$$\begin{cases} x - y - z = 0, \\ x + 4y + 2z = 0, \\ 3x + 7y + 3z = 0. \end{cases}$$

△ We evaluate the determinant of system A:

$$\det A = \det \begin{pmatrix} 1 & -1 & -1 \\ 1 & 4 & 2 \\ 3 & 7 & 3 \end{pmatrix} = \begin{vmatrix} 4 & 2 \\ 7 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = 0,$$

so that it is impossible to use Cramer's rule. To find the solutions other than the obvious one $x = 0$, $y=0$, $z=0$, we notice that the third equation can be obtained by adding together the first and doubled second equations, that is, the third equation is the consequence of the first two and therefore may be discarded. It remains to solve the system

$$\begin{cases} x - y = z, \\ x + 4y = -2z. \end{cases}$$

Setting $z = t$ (an arbitrary number), we easily find that $5y = -3z$, that is, $y = -\frac{3}{5}t$, hence $x = y + z = \frac{2}{5}t$. Thus, all the solutions of the given homogeneous system can be written in the form $x = \frac{2}{5}t$, $y = -\frac{3}{5}t$, $z = t$, where t is an arbitrary number. For $t = 0$ we get the trivial solution $x = y = z = 0$. ▲

Exercises

1. Evaluate the given determinants of the second order:

(a) $\begin{vmatrix} -1 & 7 \\ 2 & 3 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 0 \\ 174 & 8 \end{vmatrix}$; (c) $\begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix}$; (d) $\begin{vmatrix} 6 & -30 \\ -1 & 5 \end{vmatrix}$.

2. Solve the given equations and inequalities:

(a) $\begin{vmatrix} 1 & 4 \\ 3x & x+22 \end{vmatrix} = 0$; (b) $\begin{vmatrix} x & x+1 \\ -4 & x+1 \end{vmatrix} = 0$;

(c) $\begin{vmatrix} 3x-3 & 2 \\ x & 1 \end{vmatrix} > 0$; (d) $\begin{vmatrix} 1 & x+5 \\ 2 & x \end{vmatrix} < 0$.

3. Find all the solutions of the given systems of equations:

(a) $\begin{cases} 3x-5y=13, \\ 2x+7y=81; \end{cases}$ (b) $\begin{cases} 3x-4y=1, \\ 3x+4y=18; \end{cases}$ (c) $\begin{cases} x-y\sqrt{3}=1, \\ x\sqrt{3}-3y=\sqrt{3}. \end{cases}$

4. Evaluate the given determinants of the third order:

(a) $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 5 & 0 & -1 \end{vmatrix}$; (b) $\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & 16 \\ 0 & -1 & 10 \end{vmatrix}$;

(c) $\begin{vmatrix} 2 & -1 & 3 \\ -2 & 3 & 2 \\ 0 & 2 & 5 \end{vmatrix}$; (d) $\begin{vmatrix} 147 & 294 & 147 \\ 10 & 20 & 10 \\ 4 & 12 & 8 \end{vmatrix}$.

5. Find all the solutions of the given equations:

(a) $\begin{cases} x+y-z=36, \\ x+z-y=13, \\ y+z-x=7; \end{cases}$ (b) $\begin{cases} x+2y+z=4, \\ 3x-5y+3z=1, \\ 2x+7y-z=8; \end{cases}$

(c) $\begin{cases} 2x-4y+9z=28, \\ 7x+3y-6z=-1, \\ 7x+9y-9z=5; \end{cases}$ (d) $\begin{cases} x+y+z=36, \\ 2x-3z=-17, \\ 6x-5z=7. \end{cases}$

6. Find all the solutions of the given systems of equations:

(a) $\begin{cases} x+2y-4z=1, \\ 2x+y-5z=-1, \\ x-y-z=-2; \end{cases}$ (b) $\begin{cases} 2x-y+z=-2, \\ x+2y+3z=-1, \\ x-3y-2z=3. \end{cases}$

7. Find all the solutions of the given homogeneous systems of equations:

(a) $\begin{cases} 2x+y-z=0, \\ x+2y+z=0, \\ 2x-y+3z=0; \end{cases}$ (b) $\begin{cases} x+y+3z=0, \\ x+2y-2z=0, \\ -\frac{1}{2}x-y+z=0. \end{cases}$

1.5. Vector Product of Two Vectors. The Scalar Triple Product

1. The Vector Product Defined. In physics, the moment of force F about the point O is represented by the vector $\overrightarrow{OM}$ which is perpendicular to the plane containing the point O and the vector F (Fig. 23). In this case, the length of the vector $\overrightarrow{OM}$, that is, the magnitude

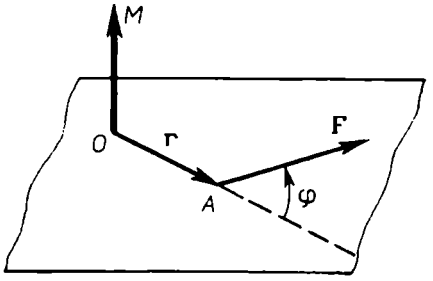

Fig. 23

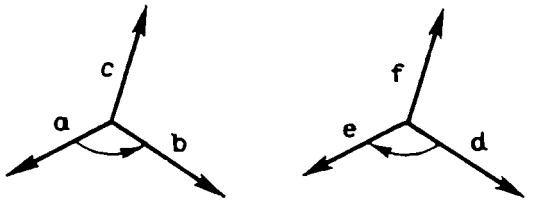

Fig. 24

of the force moment is defined as the product of the length of the radius vector $\mathbf{r} = \overrightarrow{OA}$ of the point of application of the force F by the length of the vector F and by the sine of the angle between these vectors, that is,

$$|\overrightarrow{OM}| = |\mathbf{r}|\,|\mathbf{F}|\sin\varphi,$$

where φ is the angle between the vectors **r** and **F**. Such a vector $\overrightarrow{OM}$, which is perpendicular both to the vector **r** and to the vector **F**, is called the vector product of the vector **r** by the vector **F**. Prior to defining the vector product of two arbitrary vectors **a** and **b**, let us introduce the notion of the right- and left-handed triples of vectors.

Definition 1. Three noncoplanar vectors **a**, **b**, and **c**, taken in the indicated order, form a right-handed triple if, on having been brought to a common origin, the vector **c** is found on that side from the plane containing the vectors **a** and **b** wherefrom a rotation of the vector **a** to the vector **b** (by the shortest route) is seen as performed anticlockwise. If the rotation is clockwise, then the system of three vectors is called left-handed.

The vectors **a**, **b**, **c**, shown in Fig. 24 form a right-handed system, while the vectors **d**, **e**, **f** (in the same figure) a left-handed triple. In

Fig. 23, the vectors $\mathbf{r}$, $\mathbf{F}$, $\overrightarrow{OM}$ form a right-handed triple. In Fig. 18, the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the Cartesian coordinate system also form a right-handed triple.

Definition 2. The vector (cross) product of a vector $\mathbf{a}$ by a noncollinear vector $\mathbf{b}$ is a third vector $\mathbf{c}$ which satisfies the following three conditions:

(a) its length $|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}| \sin\varphi$, where φ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$;

(b) it is perpendicular to the vectors $\mathbf{a}$ and $\mathbf{b}$, that is, $\mathbf{c} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{b}$;

(c) the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, taken in the indicated order, form a right-handed system.

Since the angle between collinear vectors is equal either to 0° or to 180°, the vector product of collinear vectors is a null vector. The vector (cross) product of the vector $\mathbf{a}$ by the vector $\mathbf{b}$ is denoted by $\mathbf{a} \times \mathbf{b}$ (alternative notation $[\mathbf{ab}]$). The conditions (a) and (b) define the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with an accuracy to two opposite directions. The condition (c) enables us to choose one of these two directions. From the geometrical point of view, the condition (a) is equivalent to the assertion that the length of the vector $\mathbf{c}$ is numerically equal to the area of the parallelogram constructed on the vectors $\mathbf{a}$ and $\mathbf{b}$ (Fig. 25).

Theorem. *For two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *to be collinear, it is necessary and sufficient that the equality*

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \tag{1}$$

is satisfied.

□ If the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear, then the angle φ between them is either 0° or 180°, that is, $\sin\varphi = 0$. Hence it follows $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, and the necessity has been proved. Let us now prove the sufficiency. Let equality (1) occur for nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. It follows from (1) that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\varphi = 0$, whence $\sin\varphi = 0$ since $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$. As a result, either $\varphi = 0°$ or $\varphi = 180°$, that is, the vectors $\mathbf{a}$ and $\mathbf{b}$ are collinear. The sufficiency has been proved. ■

Example. Let $A(0, 0)$, $B\left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$, and $C(1, 0)$ be vertices of a triangle ABC. Evaluate $|\overrightarrow{AB} \times \overrightarrow{AC}|$.

△ According to the definition of the vector product, we have

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}|\,|\overrightarrow{AC}| \sin\varphi.$$

We compute the lengths of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$: $\overrightarrow{AB} = \left\{\frac{3}{4}; \frac{\sqrt{3}}{4}\right\}$, and $|\overrightarrow{AB}| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \frac{\sqrt{3}}{2}$, $\overrightarrow{AC} = \{1, 0\}$ and

$|\overrightarrow{AC}| = \sqrt{1^2+0^2} = 1$. To find the angle φ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, we make use of the scalar product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$: $\overrightarrow{AB}\cdot\overrightarrow{AC} = \frac{3}{4}\cdot 1 + \frac{\sqrt{3}}{4}\cdot 0 = \frac{3}{4}$. On the other hand, $\overrightarrow{AB}\times\overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\cos\varphi = \frac{\sqrt{3}}{2}\cos\varphi$. Thus $\frac{3}{4} = \frac{\sqrt{3}}{2}\cos\varphi$, whence $\cos\varphi = \frac{\sqrt{3}}{2}$, that is, $\varphi = 30°$. Finally, $|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{\sqrt{3}}{2}\cdot 1 \times \sin 30° = \frac{\sqrt{3}}{4}$. ▲

2. Properties of the Vector Product.

1. For any vectors **a** and **b** the following equality is valid:

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}. \tag{1}$$

This is the anticommutative property of the vector product.

2. For any vectors **a** and **b** and any number k the following equality holds true:

$$k\mathbf{a}\times\mathbf{b} = \mathbf{a}\times k\mathbf{b} = k\cdot(\mathbf{a}\times\mathbf{b}). \tag{2}$$

Consequently, a numerical factor can be taken outside the sign of a vector product. In other words, a vector product is associative relative to a scalar multiplier.

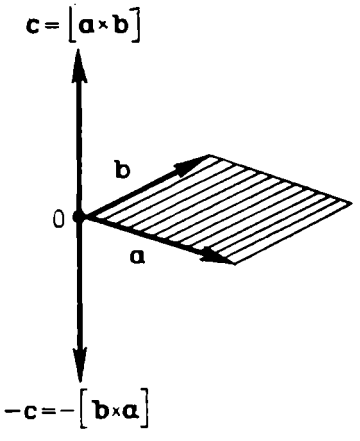

Fig. 25

3. The distributive property of a vector product is expressed by the equality

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}. \tag{3}$$

□ To prove property 1, let us note that, according to the definition of the vector product, we have

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\varphi = |\mathbf{b}|\,|\mathbf{a}|\sin\varphi = |\mathbf{b}\times\mathbf{a}|,$$

where φ is the angle between the vectors **a** and **b**. The vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ are perpendicular to the plane specified by the vectors **a** and **b**. But since the vectors **a**, **b**, and $\mathbf{a}\times\mathbf{b}$ (as also the vectors **b**, **a**, and $\mathbf{b}\times\mathbf{a}$) form right-handed triples, the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ must be in opposite directions (Fig. 25). Properties 2 and 3 are taken without proof. ■

Example. Find the length of the vector $(2\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})$ if $\mathbf{a}\perp\mathbf{b}$, $|\mathbf{a}| = 1$, and $|\mathbf{b}| = 2$.

△ By virtue of properties 2 and 3 of the vector product $(2\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{a}) - \mathbf{b}\times\mathbf{a} + 2(\mathbf{a}\times\mathbf{b}) - \mathbf{b}\times\mathbf{b}$. Since

every vector is collinear with itself, and the vector product of collinear vectors is assumed to be equal to the null vector, we have $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0$. According to property 1, we have $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$, and, finally, $(2\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 3(\mathbf{a} \times \mathbf{b})$. Consequently, $|(2\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b})| = 3|(\mathbf{a} \times \mathbf{b})| = 3|\mathbf{a}||\mathbf{b}| \sin 90^\circ = 3 \cdot 1 \cdot 2 \cdot 1 = 6$. ▲

3. The Vector Product of Two Vectors Given by Their Coordinates. Let two vectors $\mathbf{a} = \{x_a, y_a, z_a\}$, and $\mathbf{b} = \{x_b, y_b, z_b\}$ be given by their coordinates in a rectangular Cartesian coordinate system O, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the triple of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being right-handed (for definiteness, this will be always assumed in what follows). Since

$$\mathbf{a} = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k} \text{ and } \mathbf{b} = x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k},$$

using the properties of the vector product, we get:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}) \times (x_b\mathbf{i} + y_b\mathbf{j} + z_b\mathbf{k}) \\ &= x_a x_b (\mathbf{i} \times \mathbf{i}) + x_a y_b (\mathbf{i} \times \mathbf{j}) + x_a z_b (\mathbf{i} \times \mathbf{k}) + y_a x_b (\mathbf{j} \times \mathbf{i}) \\ &\quad + y_a y_b (\mathbf{j} \times \mathbf{j}) + y_a z_b (\mathbf{j} \times \mathbf{k}) + z_a x_b (\mathbf{k} \times \mathbf{i}) \\ &\quad + z_a y_b (\mathbf{k} \times \mathbf{j}) + z_a z_b (\mathbf{k} \times \mathbf{k}). \end{aligned} \tag{1}$$

By the definition of the vector product, we find $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$; $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$; $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j}$; $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$. Now formula (1) takes the form:

$$\mathbf{a} \times \mathbf{b} = x_a y_b \mathbf{k} - x_a z_b \mathbf{j} - y_a x_b \mathbf{k} + y_a z_b \mathbf{i} + z_a x_b \mathbf{j} - z_a y_b \mathbf{i} \text{ or}$$

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b)\mathbf{i} + (z_a x_b - x_a z_b)\mathbf{j} + (x_a y_b - y_a x_b)\mathbf{k}. \tag{2}$$

Formula (2) is awkward and inconvenient for memorizing. Here, we may resort to the help of the determinants studied in the preceding section. Indeed, it is easy to see that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} \mathbf{k}. \tag{3}$$

Formula (3) can be written still briefer:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix}, \tag{4}$$

since the right-hand side of equality (3) corresponds to the formal expansion of a third-order determinant in elements of the first row.

Example 1. Find the vector product $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a} = \{1, 2, -1\}$ and $\mathbf{b} = \{5, 4, 3\}$.

△ Using formula (4), we compute: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 5 & 4 & 3 \end{vmatrix} =$

$$\mathbf{i}\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ 5 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = 10\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}. \text{ ▲}$$

Example 2. Find the area of the triangle with vertices at the points A (0, 1), B (4, 5), and C (6, —1).

△ The given triangle is constructed on the vectors $\overrightarrow{AB} = \{4 - 0, 5 - 1\} = \{4, 4\}$ and $\overrightarrow{AC} = \{6 - 0, -1 - 1\} = \{6, -2\}$.

Then the area S of the triangle ABC is computed by the formula

$$S = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|.$$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ can be regarded as three-dimensional vectors having a zero z-coordinate (the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in the xy-plane and have zero projections on the z-axis), and so we may write: $\overrightarrow{AB} = \{4, 4, 0\}$ and $\overrightarrow{AC} = \{6, -2, 0\}$. We then compute the vector product:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 6 & -2 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 4 & 0 \\ -2 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 4 \\ 6 & -2 \end{vmatrix} = -32\mathbf{k}.$$

Consequently, $S = \frac{1}{2} \cdot 32 = 16$. ▲

4. The Scalar Triple Product (Box Product) and its Geometrical Meaning. For two vectors **a** and **b**, we have defined two types of multiplication: scalar (this operation results in a scalar, that is, in a number) and vectorial (this operation yields a vector).

When three vectors are multiplied in turn, the following constructions are possible: $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$, $((\mathbf{a} \times \mathbf{b}) \times \mathbf{c})$, and $(\mathbf{a} \times \mathbf{b})\,\mathbf{c}$.

The first of the constructions corresponds to a trivial case of multiplying a scalar $\mathbf{a} \cdot \mathbf{b}$ by a vector **c** and is of no specific interest. The second construction called the double vector product often appears in the process of vector computations, but is not aimed at concrete applications.

The last of the above constructions represents a scalar obtained as a result of scalar multiplication of the vector $\mathbf{a} \times \mathbf{b}$ by the vector **c** and is called the scalar triple product of the vectors **a**, **b**, and **c**. It is denoted by $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. This construction has a clear geometrical interpretation and is widely used in practical problems.

Theorem. *The absolute value of the scalar triple product of three noncoplanar vectors* **a**, **b**, *and* **c** *is equal to the volume of the parallelepiped constructed on the given vectors.*

□ As is known (see subsection 1 of this section), the absolute value of the vector (cross) product $|\mathbf{a} \times \mathbf{b}|$ is equal to the area S of the parallelogram constructed on the vectors **a** and **b**. Then,

the definition of the scalar triple product imply the equality

$$(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = |\mathbf{a} \times \mathbf{b}||\mathbf{c}|\cos\varphi = S|\mathbf{c}|\cos\varphi, \qquad (1)$$

where φ is the angle between the vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$. On the other hand, the volume V of the parallelepiped constructed on the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (Fig. 26) is equal to the product of the area of its base S by the altitude h, where $h = |\overrightarrow{AA_2}| = |\mathbf{c}|\cos\varphi|$. Thus,

$$V = Sh = S|\mathbf{c}||\cos\varphi|. \qquad (2)$$

Comparing (1) and (2), we find that

$$\mathbf{V} = |(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}|. \blacksquare \qquad (3)$$

From formula (1) it is seen that the sign of the scalar triple product of three vectors coincides with that of $\cos\varphi$. Therefore the scalar triple product is positive when the vector $\mathbf{c}$ is in the same direction from the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$ as the vector $\mathbf{a} \times \mathbf{b}$ (that is, the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-handed triple), and $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = V$. The scalar triple product is negative when the

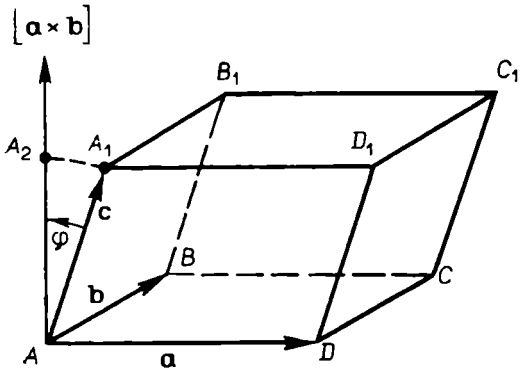

Fig. 26

vectors $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ are in opposite directions from the plane in which the vectors $\mathbf{a}$ and $\mathbf{b}$ lie (in this case, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a left-handed triple), and $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = -V$.

Example 1. Compute $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}$ if the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right-handed triple, $\mathbf{a} \perp \mathbf{c}$, $\mathbf{b} \perp \mathbf{c}$, $|\mathbf{a}| = |\mathbf{b}| = 2$, $|\mathbf{c}| = 5$, and the angle φ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to 120°.

△ Since $\mathbf{c} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{b}$, the vector $\mathbf{c}$ is perpendicular to the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$, and, hence the vector $\mathbf{c}$ is collinear with the vector $\mathbf{a} \times \mathbf{b}$. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a right-handed triple, the vector $\mathbf{c}$ is in the same direction as the vector $\mathbf{a} \times \mathbf{b}$. Thus, the angle between the vectors $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is equal to 0°. Further, by definition, $(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c} = |\mathbf{a}||\mathbf{b}|\sin 120°|\mathbf{c}|\cos 0° = 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} \times 5 \cdot 1 = 10\sqrt{3}$. ▲.

Example 2. Compute $((\mathbf{i}+\mathbf{j})\times(\mathbf{j}-\mathbf{i}))\,\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are mutually perpendicular unit vectors forming a right-handed triple.

△ Applying the properties of the vector and scalar products, we find: $((\mathbf{i}+\mathbf{j})\times(\mathbf{j}-\mathbf{i}))\cdot\mathbf{k} = ((\mathbf{i}\times\mathbf{j})-(\mathbf{i}\times\mathbf{i})+(\mathbf{j}\times\mathbf{j})-(\mathbf{j}\times\mathbf{i}))\cdot\mathbf{k} = 2\,(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k} = 2$, since $(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k} = V = 1$. ▲

Let now the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be given by their Cartesian coordinates:

$$\mathbf{a} = \{x_a, y_a, z_a\}, \quad \mathbf{b} = \{x_b, y_b, z_b\}, \text{ and } \mathbf{c} = \{x_c, y_c, z_c\}.$$

We multiply scalarwise the vector $\mathbf{a}\times\mathbf{b}$ (see formula (3) from the preceding subsection) by the vector $\mathbf{c} = x_c\mathbf{i} + y_c\mathbf{j} + z_c\mathbf{k}$: $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} =$

$$x_c\begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} - y_c\begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} + z_c\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} = \begin{vmatrix} x_c & y_c & z_c \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix}.$$

The third-order determinant remains unchanged if its rows are interchanged twice (see subsection 2 of 1.4). Finally, we find that

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}, \tag{4}$$

that is, the scalar triple product of three vectors is equal to the determinant of the third order whose first row consists of the coordinates of the first vector, the second row of the coordinates of the second vector, and the third row of the coordinates of the third vector.

The theorem proved in the present subsection implies that the vectors $\mathbf{a} = \{x_a, y_a, z_a\}$, $\mathbf{b} = \{x_b, y_b, z_b\}$ and $\mathbf{c} = \{x_c, y_c, z_c\}$ are coplanar if and only if their scalar triple product is equal to zero:

$$\begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix} = 0. \tag{5}$$

Example 3. Find out whether the vectors $\mathbf{a} = \{4, 2, 1\}$, $\mathbf{b} = \{8, 6, 3\}$ and $\mathbf{c} = \{5, 2, 1\}$ are coplanar.

△ Let us take advantage of coplanarity condition (5):

$$\begin{vmatrix} 4 & 2 & 1 \\ 8 & 6 & 3 \\ 5 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 6 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 8 & 3 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 8 & 6 \\ 5 & 2 \end{vmatrix} = 0 + 14 - 14 = 0.$$

Consequently, the given vectors are coplanar. ▲

Example 4. Find the volume of the parallelepiped constructed on the vectors $\mathbf{a} = \{-2, -1, -1\}$, $\mathbf{b} = \{4, 3, 1\}$, and $\mathbf{c} = \{1, 2, 3\}$.

△ We find the scalar triple product of the given vectors:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} -2 & -1 & -1 \\ 4 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -14 + 11 - 5 = -8.$$

The volume of the parallelepiped is equal to the absolute value of the scalar triple product: $V = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |-8| = 8$. ▲

Exercises

1. Find out whether the vectors **a**, **b**, and **c** form a right-handed or a left-handed triple if **i**, **j**, **k** form a right-handed triple: (a) $\mathbf{a} = -\mathbf{i}, -\mathbf{j}, \mathbf{b} = \mathbf{j}, \mathbf{c} = \mathbf{k}$; (b) $\mathbf{a} = \mathbf{i} - \mathbf{j}, \mathbf{b} = \mathbf{j}, \mathbf{c} = \mathbf{i} + \mathbf{j}$.

2. Find and represent the vector $\mathbf{a} \times \mathbf{b}$: (a) $\mathbf{a} = 3\mathbf{i}, \mathbf{b} = 4\mathbf{j}$; (b) $\mathbf{a} = 2\mathbf{i} - \mathbf{k}, \mathbf{b} = 3\mathbf{k}$.

3. Find the area of the parallelogram constructed on the vectors $\mathbf{a} = \{2, 3\}$ and $\mathbf{b} = \{2, -1\}$.

4. Given vectors: $\mathbf{a} = \{1, -1, 1\}$, $\mathbf{b} = \{1, 2, -1\}$, $\mathbf{c} = \{0, 1, 2\}$, and $\mathbf{d} = \{2, 1, -1\}$. Compute: (a) $\mathbf{a} \times \mathbf{b}$; (b) $\mathbf{a} \times \mathbf{c}$; (c) $\mathbf{b} \times \mathbf{c}$; (d) $\mathbf{a} \times \mathbf{d}$; (e) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$; (f) $(\mathbf{a} - \mathbf{b}) \times (\mathbf{d} + \mathbf{c})$; (g) $(2\mathbf{a} + \mathbf{b}) \times (\mathbf{c} - \mathbf{d})$.

5. Find the area of the triangle with vertices at the points A (0, 1, 2), B (3, 0, 0), and C (0, 5, 0).

6. Given the vertices of a parallelogram: A (1, −2), B (−2, 2), C (4, 10), and D (7, 6). Compute its area and altitudes.

7. Vectors **a**, **b**, and **c**, forming a right-handed triple, are mutually perpendicular. Knowing that $|\mathbf{a}| = 6$, $|\mathbf{b}| = 4$, and $|\mathbf{c}| = 5$, evaluate $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

8. Find the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ of the vectors $\mathbf{a} = \{0, 1, 2\}$, $\mathbf{b} = \{4, 0, 0\}$, and $\mathbf{c} = \{9, -1, 1\}$.

9. Find out whether the vectors $\mathbf{a} = \{8, 5, -13\}$, $\mathbf{b} = \{-4, 2, 8\}$, and $\mathbf{c} = \{4, 7, -4\}$ are coplanar. If the vectors are noncoplanar, then what triple (right-handed or left-handed) do they form?

10. Find the volume of the parallelepiped constructed on the vectors: (a) $\mathbf{a} = \{1, 2, 3\}$, $\mathbf{b} = \{-1, 3, 4\}$, $\mathbf{c} = \{2, 5, 2\}$; (b) $\mathbf{a} = \{-5, -1, -5\}$, $\mathbf{b} = \{1, 1, 1\}$, $\mathbf{c} = \{1, 2, 2\}$.

11. Determine whether the given four points are coplanar: (a) A (5, 2, −2), B (6, −3, 1), C (0, 4, 3), D (2, 0, −4); (b) A (3, 5, 1), B (2, 4, 7), C (1, 5, 3), D (4, 4, 5).

Chapter 2

ANALYTIC GEOMETRY IN THE PLANE

2.1. Lines in the Plane

1. Equations of Lines in the Plane. As is known, if there is a Cartesian coordinate system in the plane, then any point of the plane is identically defined by its coordinates. In order to specify a set of points in the plane which satisfy a certain condition, it suffices to reformulate this condition in terms of coordinates. For instance, the condition: a point M with coordinates x, y is at a distance $R > 0$ from the point M_0 with coordinates x_0, y_0 is equivalent to the condition

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \tag{1}$$

This condition can be considered as an equation in two variables which defines the set of all points M of the plane situated at a distance R from the given point M_0 (x_0, y_0). Since this set is a circle of radius R with centre at the point M_0, equation (1) is said to be the equation of this circle.

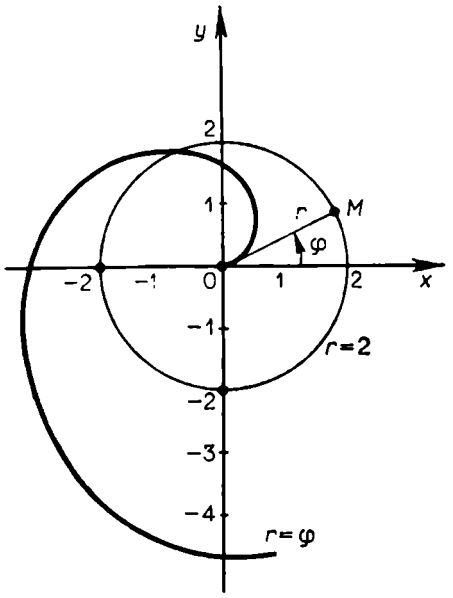

Fig. 27

From a school course of mathematics it is known that the equation $y = x^2$ specifies a plane curve which is called the parabola.

Analytic geometry in the plane studies the properties of various lines given by certain equations in two variables represented by the coordinates of arbitrary points of the line under consideration.

2. Polar Coordinate System. Along with the Cartesian coordinate system in the plane, we shall also consider a so-called polar coordinate system.

The position of an arbitrary point M in the plane (Fig. 27) will be defined if a pair of numbers r and φ corresponding to this point are given, where r is the distance from the origin O to the point M, that is, the length of the radius vector $\overrightarrow{OM}$, and φ the angle through which we have to turn the x-axis anticlockwise until it is brought into coincidence with the straight line on which the vector

$\overrightarrow{OM}$ lies. The numbers r and φ are called the polar coordinates of the point M: r is the polar radius, φ being the polar angle. In this case, the x-axis is called the polar axis, and the point O the pole. If the point M is given by the polar coordinates r and φ, then we write $M(r, \varphi)$.

To get all points of the plane, it suffices to restrict the polar angle by the interval $0 \leqslant \varphi < 360°$, however, there are problems, where such restriction is undesirable.

Let us establish a relation between polar and Cartesian coordinates of one and the same point M in the plane. It is easy to see (Fig. 27) that

$$x = r \cos \varphi, \quad y = r \sin \varphi. \tag{1}$$

Formulas (1) enable us to find Cartesian coordinates of a point from its polar coordinates. The following formulas are derived from formulas (1):

$$r = \sqrt{x^2 + y^2}. \quad \cos \varphi = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin \varphi = \frac{y}{\sqrt{x^2+y^2}}, \tag{2}$$

enabling us to change from Cartesian to polar coordinates.

Example. Find the polar coordinates of the point $M(-1, 1)$.

△ We compute by formulas (2): $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\cos \varphi = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$, $\sin \varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Hence $\varphi = 135°$. Thus $M(\sqrt{2}, 135°)$. ▲

In polar coordinates, the equation $r = 2$ specifies a circle of radius 2 centred at the point O, since this is a locus of points found at the same distance $r = 2$ from the point O. In Cartesian coordinates the same curve is described by a more complicated equation

$$x^2 + y^2 = 4.$$

The curve $r = \varphi$ (here the angle φ is not bounded from above) is called the Archimedes spiral and has the shape of an open untwisting spiral (see Fig. 27). The equation of the Archimedes spiral in Cartesian coordinates will be much more complicated.

Exercises

1. What line is given by the equation $x^2 + y^2 = 25$? Construct the graph of this equation and mark on it the points whose abscissas are equal to -5, -2, 0, 3, 5.

2. What lines are defined by the following equations (sketch them):
(a) $x - y = 0$, (b) $x + y = 0$; (c) $x + 3 = 0$; (d) $y + 5 = 0$;
(e) $x^2 - y^2 = 0$; (f) $xy = 0$; (g) $y^2 - 9 = 0$; (h) $y = |x|$;
(i) $y + |x| = 0$; (j) $y = |x + 2|$?

3. Determine which of the given lines pass through the origin:
(a) $2x + y = 0$; (b) $x^2 + y^2 - 25 = 0$;
(c) $x^2 + y^2 + 3x - 5y - 7 = 0$.

4. Find the points of intersection of the given lines with x- and y-axes:
(a) $x^2 + y^2 = 36$; (b) $(x + 4)^2 + (y - 3)^2 = 9$;
(c) $x^2 + 2y^2 - 5x + 3y + 11 = 0$.

5. Find the points of intersection of the given two lines:
(a) $x^2 + y^2 = 9$ and $x - y = 0$;
(b) $x^2 + y^2 - 16x + 4y + 18 = 0$ and $x + y = 0$.

2.2. The Straight Line

1. The Equation of a Straight Line Through Two Given Points (Two-point Form). Intercept Form of the Equation of a Straight Line. As is known, we can draw a unique straight line through two given points. Let us find the equation of this straight line if the Cartesian coordinates of two points M_1 and M_2 given in the plane are known.

Let the points M_1 and M_2 have coordinates x_1, y_1, and x_2, y_2, respectively. Let x, y denote the coordinates of an arbitrary point M taken on the straight line l under consideration. The point $M(x, y)$ lies on the straight line l if and only if the vectors $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1\}$ and $\overrightarrow{M_1M} = \{x - x_1, y - y_1\}$ are collinear (Fig. 28), that is, if the equality

$$\overrightarrow{M_1M} = \lambda \cdot \overrightarrow{M_1M_2} \tag{1}$$

is valid. Here λ is a certain number (see subsection 5 from 1.1). From the vector equality given by (1) there follow the equalities of like coordinates

$$x - x_1 = \lambda (x_2 - x_1) \text{ and } y - y_1 = \lambda (y_2 - y_1),$$

whence

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \tag{2}$$

This is the equation of a straight line passing through two given points. Equation (2) holds true when $x_2 - x_1 \neq 0$ and $y_2 - y_1 \neq 0$.

If, for instance, the straight line l is parallel to the y-axis, then its equation takes the form:

$$x - x_1 = 0 \text{ or } x = x_1,$$

the y-coordinate being arbitrary.

Example 1. Given three vertices of a triangle: A (0, 0), B (2, 1), and C (—1, 1). Write the equations of its sides.

△ The equations of the sides of a triangle are found by formula (2). For the side AB equation (2) takes the form:

$$\frac{x-x_A}{x_B-x_A} = \frac{y-y_A}{y_B-y_A},$$

where x_A, y_A and x_B, y_B are coordinates of the points A and B,

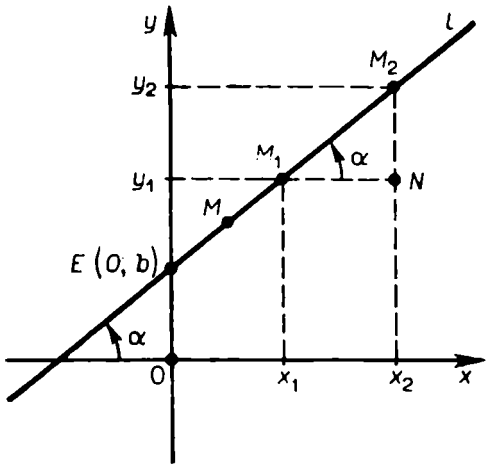

Fig. 28

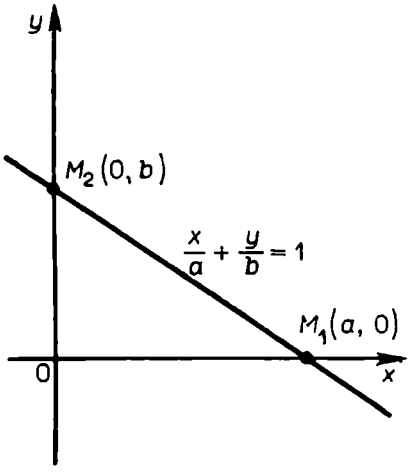

Fig. 29

respectively. Substituting the numerical values for the coordinates of the vertices A and B, we find

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} \quad \text{or} \quad x-2y=0.$$

Similarly, we find that the equation of the side AC is

$$\frac{x-0}{-1-0} = \frac{y-0}{1-0} \quad \text{or} \quad x+y=0.$$

The side BC is parallel to the x-axis since $y_B = y_C = 1$. The equation of the side BC has the form:

$$y = 1,$$

the x-coordinate being arbitrary. ▲

Consider a particular case when two points M_1 and M_2 specifying the straight line l lie on the coordinate axes Ox and Oy. Let, for definiteness, the point M_1 $(a, 0)$ lies on the x-axis, and the point M_2 $(0, b)$ on the y-axis (Fig. 29). In this case the equation of the

straight line given by (2) takes the form:

$$\frac{x-a}{0-a}=\frac{y-0}{b-0},$$

or

$$\frac{x}{a}+\frac{y}{b}=1. \tag{3}$$

Equation (3) is the intercept form of the equation of a straight line since the numbers a and b indicate the line segments intercepted by the straight line on the coordinate axes.

Example 2. Compute the area of the triangle cut off by the straight line $\frac{x}{11}-\frac{y}{7}=1$ from the coordinate angle.

△ If the straight line is given by the intercept form (3) of its equation, then the area S of the triangle cut off by this straight line from the coordinate angle (see Fig. 29) is computed by the formula $S=\frac{1}{2}|ab|$. Since for the given straight line $a=11$ and $b=-7$, we have $S=\frac{77}{2}$. ▲

2. The Slope of a Straight Line. For a sloping straight line (when $x_2 \neq x_1$) let us write equation (2) of the preceding subsection in the form

$$y-y_1=k(x-x_1), \tag{1}$$

where

$$k=\frac{y_2-y_1}{x_2-x_1}. \tag{2}$$

From the triangle M_1M_2N we find (see Fig. 28) that the slope k is equal to $\tan\alpha$, where α is the angle through which it is necessary to rotate the x-axis anticlockwise to bring the x-axis into coincidence with the straight line l. This angle is called the angle of inclination of a straight line to the x-axis, and the coefficient k is termed the slope of the straight line. In other words, the tangent of the angle of inclination of a straight line to the x-axis is called the slope of the line. Equation (1) is called the equation of a straight line through a given point $M_1(x_1, y_1)$ with a given slope k.

Example 1. Find the angle of inclination of a straight line passing through the points $M_1(1, -1)$ and $M_2(-2, 2)$.

△ The equation of the straight line passing through two given points has the form $\frac{x-1}{-2-1}=\frac{y+1}{2+1}$, or $y=-x$. Here $k=\tan\alpha=-1$. Consequently, $\alpha=135°$. ▲

Example 2. Write the equation of the straight line passing through the point $M_1(-1, 3)$ and making an angle of 45° with the x-axis.

△ The slope of the desired straight line is $k = \tan 45° = 1$. Substituting the coordinates $x_1 = -1$, $y_1 = 3$ into equation (1), we find the equation of the straight line: $y - 3 = 1 \cdot (x + 1)$ or $x - y + 4 = 0$. ▲

3. The Equation of a Straight Line Passing Through a Given Point Parallel to a Given Vector. Let a point $M_1(x_1, y_1)$ and a vector $\mathbf{t} = \{a, b\}$ be given in a Cartesian coordinate system. It is required to draw through the point M_1 a straight line l parallel to the vector $\mathbf{t}$ which is called the direction vector of the straight line. The point $M(x, y)$ lies on the straight line l if and only if the vector $\overrightarrow{M_1M}$ is collinear with the vector $\mathbf{t}$. In this case, the equality $\overrightarrow{M_1M} = \lambda\mathbf{t}$, where λ is a number (see subsection 5 from 1.1). The vector equality implies the equalities of like coordinates: $x - x_1 = \lambda a$ and $y - y_1 = \lambda b$. Hence we find the required equation of the straight line:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}. \tag{1}$$

Equation (1) is called the canonical equation of the straight line with direction vector $\mathbf{t} = \{a, b\}$.

Example 1. Write the canonical equation of the straight line passing through the point $M_1(1, 2)$ parallel to the vector $\mathbf{t} = \{4, 3\}$.

△ Substituting the coordinates of the point M_1 and those of the direction vector $\mathbf{t}$ into equation (1), we get $\frac{x-1}{4} = \frac{y-2}{3}$. ▲

Example 2. Write the equation of the straight line passing through the point $M_1(-1, 7)$ parallel to the vector $\mathbf{t} = \{9, 0\}$.

△ Since the second coordinate of the direction vector is equal to zero, that is, the vector is parallel to the x-axis, the equation of the straight line is $y - 7 = 0$ or $y = 7$, the x-coordinate being arbitrary. ▲

4. The Equation of a Straight Line Passing Through a Given Point Perpendicular to a Given Vector. Let a point $M_1(x_1, y_1)$ and a vector $\mathbf{N} = \{A, B\}$ be given. It is required to draw through the point M_1 a straight line l perpendicular to the vector $\mathbf{N}$. A point $M(x, y)$ lies on the straight line l if and only if the vector $\overrightarrow{M_1M} = \{x - x_1, y - y_1\}$ is perpendicular to the vector $\mathbf{N}$, that is, when the equality

$$\mathbf{N} \cdot \overrightarrow{M_1M} = 0 \tag{1}$$

is valid. Since the scalar product of two vectors is equal to the sum of the products of like coordinates of these vectors, equation (1) takes the form:

$$A(x - x_1) + B(y - y_1) = 0. \tag{2}$$

Equation (2) is called the equation of a straight line passing through a given point $M_1(x_1, y_1)$ perpendicular to a given vector $\mathbf{N} = \{A, B\}$. The vector $\mathbf{N}$ is said to be the normal vector of the straight line.

Example 1. Write the equation of the straight line passing through the point M_1 (1, —1) perpendicular to the vector $\mathbf{N} = \{-1, 1\}$ (Fig. 30).

△ Using equation (2), we find the equation of the straight line: $(-1)\cdot(x - 1) + 1\cdot(y + 1) = 0$ or, finally, $x - y - 2 = 0$. ▲

Example 2. Write the equation of the straight line passing through the point M_1 (7, 8) perpendicular to the x-axis.

△ Since the required straight line is perpendicular to the x-axis, we may take the vector $\mathbf{i} = \{1, 0\}$ for the normal vector of this line. Substituting the coordinates of the point M_1 and those of the normal vector $\mathbf{N} = \mathbf{i}$ into equation (2), we find the desired equation: $1 \cdot (x - 7) + 0 \cdot (y - 8) = 0$ or $x = 7$. ▲

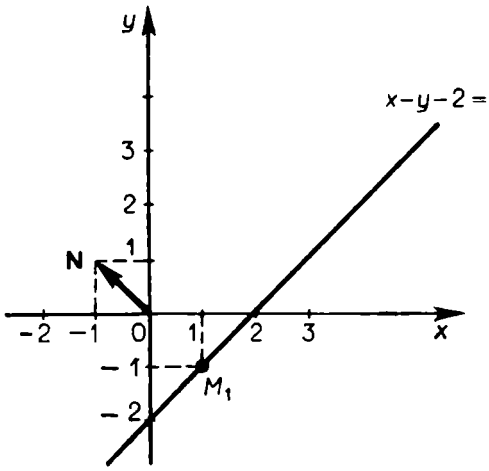

Fig. 30

5. The General Equation of the Straight Line in the Plane. Various notations of the equation of the straight line considered in the previous subsections have, in principle, the same form:

$$Ax + By + C = 0, \qquad (1)$$

where A, B, and C are some numbers. Equation (1) is called the general equation of the straight line.

We can easily see that if a straight line is given by the general equation, then the normal vector of this straight line is the vector $\mathbf{N} = \{A, B\}$ (and any vector $\lambda\mathbf{N} = \{\lambda A, \lambda B\}$, where λ is a number), and the direction vector of this straight line is the vector $\mathbf{t} = \{-B, A\}$.

If $C = 0$, then straight line (1) passes through the origin, if $B = 0$, then the straight line is parallel to the y-axis, and if $A = 0$, then it is parallel to the x-axis.

Example 1. Indicate the normal vectors for the straight lines given by the equations $x - y - 2 = 0$, $y = 2x + 1$, and $x = 7$.

△ The normal vector of the first straight line is the vector $\mathbf{N}_1 = \{1, -1\}$, of the second, the vector $\mathbf{N}_2 = \{2, -1\}$, and of the third the vector $\mathbf{N}_3 = \{1, 0\}$. ▲

Example 2. A straight line is drawn through the point of intersection of the straight lines $5x - 5y - 1 = 0$ and $5x + 6y - 1 = 0$ perpendicular to the straight line $x - y + 92 = 0$. Write its equation.

△ The coordinates x_1, y_1 of the point of intersection of the two straight lines are found from the system of equations

$$\begin{cases} 5x_1 - 5y_1 - 1 = 0, \\ 5x_1 + 6y_1 - 1 = 0, \end{cases}$$

whence $x_1 = \frac{1}{5}$, $y_1 = 0$. For the direction vector $\mathbf{t}$ of the desired straight line we may take the normal vector of the straight line $x - y + 92 = 0$, that is, we may set $\mathbf{t} = \{1, -1\}$. It remains only to write the canonical equation of the desired straight line:

$$\frac{x - \frac{1}{5}}{1} = \frac{y - 0}{-1}.$$

The general equation of the required straight line has the form $5x + 5y - 1 = 0$. ▲

Exercises

1. Write the equation of the straight line passing through two given points if: (a) M_1 (−1, 5) and M_2 (−3, −2); (b) O (0, 0) and M (4, −3).

2. For the straight line $y = -\frac{3}{4}x + 7$ write its equation in intercept form.

3. For the straight line $2x - 9y = 7$ write its equation in intercept form.

4. Find the points of intersection of the given straight line with the coordinate axes and construct this line: (a) $2x - 3y - 12 = 0$; (b) $4x - 6y + 12 = 0$.

5. Given successively the vertices of a convex quadrilateral: A (−6, −2), B (6, 7), C (9, 3), and D (1, −3). Determine the point of intersection of its diagonals.

6. Compute the area of the triangle cut off by the straight line $4x + 3y - 12 = 0$ from the coordinate angle.

7. Write the equation of the straight line passing through the point A (0, 6) if the area of the triangle bounded by the straight line and coordinate axes is equal to 16.

8. Which of the straight lines $3x - 4y + 5 = 0$ or $x - y = 0$ intercepts a greater segment on the axis of ordinates?

9. Construct the straight line passing through the point A (3, −2) perpendicular to the vector $\mathbf{N} = \{1, 4\}$.

10. A straight line is given by the equation $-4(x+3)+\frac{1}{5}(y-2)=0$. Indicate some point belonging to the line and its normal vector.

11. Among straight lines $A(x+2)+B(y+3)=0$ single out the one which is perpendicular to the vector $\mathbf{N}=\{4, 1\}$.

12. Construct the straight line passing through the point $D(0, 4)$ perpendicular to the vector $\mathbf{j}$.

13. Through the points of intersection of the straight line $5x-2y-10=0$ with the coordinate axes perpendiculars are drawn to this line. Write their equations.

14. Write the general equation of the given straight line and indicate the coordinates of the normal vector:

(a) $\frac{x-2}{5}=\frac{y+1}{3}$; (b) $\frac{x}{2}+\frac{y}{3}=1$; (c) $-\frac{1}{2}(x+7)+3\left(y-\frac{4}{5}\right)=0$.

15. Construct the following straight lines: (a) $2x-y+6=0$; (b) $x-2y-3=0$; (c) $x+y+1=0$.

16. Find the points of intersection of the following pairs of straight lines: (a) $3x-2y-1=0$ and $4x+y-11=0$; (b) $x-y-7=0$ and $3x-3y+5=0$.

17. Given the equations of the sides of a triangle: $x-y+4=0$, $4x+2y-19=0$, and $5x+6y+9=0$. Find the coordinates of its vertices.

18. Write the equations of the straight lines passing through the point $M(3, -4)$ and (a) parallel to the coordinate axes; (b) perpendicular to the coordinate axes.

19. Find the tangent of the angle of inclination of the straight line $2x-5y+13=0$ and determine the segment it intercepts on the axis of ordinates.

20. Through the point of intersection of the straight lines $x-y+4=0$ and $4x+2y-19=0$ a straight line is drawn parallel to the straight line $2x-3y+6=0$. Write its equation.

21. Through the point of intersection of the straight lines $4x+2y-19=0$ and $5x+6y+6=0$ a straight line is drawn perpendicular to the straight line $x+y+1=0$. Write its equation.

22. Given a straight line $\frac{x}{5}+\frac{y}{6}=1$. Write the equation of the straight line passing through the point of intersection of this line and the axis of abscissas perpendicular to the bisector of the first quadrant.

2.3. Solving Some Problems in Plane Geometry

1. Computing the Angle Between Two Straight Lines. The Conditions of Parallelism and Perpendicularity of Two Straight Lines.

Let two straight lines l_1 and l_2 be given by the equations

$$A_1x + B_1y + C_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2 = 0. \tag{1}$$

Let ψ denote the angle between the straight lines l_1 and l_2 (it lies in the interval from 0° to 90°), and let φ denote the angle between their normal vectors $\mathbf{N}_1 = \{A_1, B_1\}$ and $\mathbf{N}_2 = \{A_2, B_2\}$ which lies

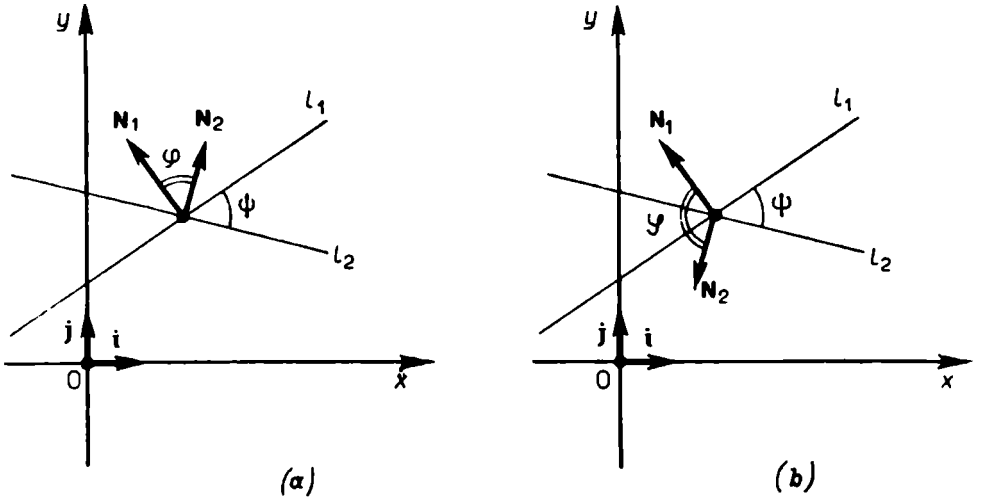

Fig. 31

in the interval from 0° to 180°. If $\varphi \leqslant 90°$ (Fig. 31, *a*), then $\psi = \varphi$, and if $\varphi > 90°$ (Fig. 31, *b*), then $\psi = 180° - \varphi$. It is obvious that in both cases the following equality is valid:

$$\cos\psi = |\cos\varphi|.$$

Since the definition of the scalar product of two vectors implies that

$$\cos\varphi = \frac{\mathbf{N}_1\mathbf{N}_2}{|\mathbf{N}_1|\,|\mathbf{N}_2|},$$

we have

$$\cos\psi = \frac{|\mathbf{N}_1\mathbf{N}_2|}{|\mathbf{N}_1|\,|\mathbf{N}_2|}. \tag{2}$$

Writing the right-hand side of formula (2) in terms of the coordinates of the normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$, we get the desired expression for the cosine of the angle between the straight lines l_1 and l_2:

$$\cos\psi = \frac{|A_1A_2 + B_1B_2|}{\sqrt{A_1^2+B_1^2}\,\sqrt{A_2^2+B_2^2}}. \tag{3}$$

Example 1. Compute the angle ψ between the straight lines $x - 2y + 1 = 0$ and $2x + y - 3 = 0$.

△ By formula (3), we find:

$$\cos\psi = \frac{|1\cdot 2 - 2\cdot 1|}{\sqrt{1^2+(-2)^2}\sqrt{2^2+1^2}} = 0,$$

that is, $\psi = 90°$, and the straight lines are mutually perpendicular. ▲

For straight lines (1) to be perpendicular, it is necessary and sufficient that

$$A_1A_2 + B_1B_2 = 0 \tag{4}$$

since only in this case $\cos\psi = 0$.

For straight lines (1) to be parallel, it is necessary and sufficient that their normal vectors be collinear, that is, that the equality $\mathbf{N}_2 = \lambda\mathbf{N}_1$ be satisfied, where λ is some number (see 1.1). From the vector equality there follow the equalities of like coordinates: $A_2 = \lambda\cdot A_1$ and $B_2 = \lambda\cdot B_1$. Hence we find the condition for parallelism of straight lines (1):

$$\frac{A_1}{A_2} = \frac{B_1}{B_2}. \tag{5}$$

Example 2. Among the following pairs of straight lines

$$6x - 15y + 7 = 0 \quad \text{and} \quad 10x + 4y - 1 = 0,$$
$$5x - 7y - 4 = 0 \quad \text{and} \quad 3x + 2y - 13 = 0,$$
$$x - 2y + 1 = 0 \quad \text{and} \quad 2x - 4y - 1 = 0$$

indicate parallel or mutually perpendicular lines.

△ For the first pair of straight lines perpendicularity condition (4) is satisfied: $A_1A_2 + B_1B_2 = 6\cdot 10 - 15\cdot 4 = 0$. For the second pair: $A_1A_2 + B_1B_2 = 5\cdot 3 - 7\cdot 2 = 1 \neq 0$, $\frac{5}{3} \neq \frac{-7}{2}$, and the straight lines are neither perpendicular nor parallel. For the third pair of straight lines parallelism condition (5) is observed since $\frac{1}{2} = \frac{-2}{-4}$. ▲

Note that two distinct straight lines intersect if and only if they are not parallel. Thus, for two distinct straight lines to intersect, it is necessary and sufficient that

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2}. \tag{6}$$

Example 3. Prove that the straight lines $y = -x + 13$ and $y = x - 29$ intersect.

△ We write the equations of the given straight lines in the form: $x + y - 13 = 0$ and $x - y - 29 = 0$. Since $\frac{1}{1} \neq \frac{1}{-1}$, the given lines are not parallel. Hence it follows that they intersect. ▲

2. Computing the Distance from a Given Point to a Given Straight Line. Let a straight line l be given by the equation $Ax + By + C = 0$. The distance d from some point in the plane $M(x_M, y_M)$ to the straight line l is equal to the absolute value of the projection of the vector $\overrightarrow{M_1M} = \{x_M - x_1, y_M - y_1\}$, where $M_1(x_1, y_1)$ is a point on l, on the direction of the normal vector $\mathbf{N} = \{A, B\}$ (Fig. 32). Since $\mathbf{N} \cdot \overrightarrow{M_1M} = |\mathbf{N}|\,|\overrightarrow{M_1M}| \cos\varphi = |\mathbf{N}| \cdot \mathrm{pr}_n M_1M$, the desired distance d is computed by the formula

$$d = |\mathrm{pr}_n \overrightarrow{M_1M}| = \frac{|\mathbf{N} \cdot \overrightarrow{M_1M}|}{|\mathbf{N}|} = \frac{|A \cdot (x_M - x_1) + B(y_M - y_1)|}{\sqrt{A^2 + B^2}}$$

$$= \frac{|Ax_M + By_M - (Ax_1 + By_1)|}{\sqrt{A^2 + B^2}}.$$

Since the point $M_1(x_1, y_1)$ belongs to the straight line l, we have $Ax_1 + By_1 = -C$. Consequently,

$$d = \frac{|Ax_M + By_M + C|}{\sqrt{A^2 + B^2}}. \tag{1}$$

Example 1. Given a straight line $\frac{x}{2} + \frac{y}{3} = 1$. Find the distance d from this line to the origin.

△ The straight line is given by the equation $\frac{1}{2}x + \frac{1}{3}y - 1 = 0$, that is, $A = \frac{1}{2}$, $B = \frac{1}{3}$, and $C = -1$. It is required to find the distance from the given line to the point $O(0, 0)$. The desired distance d is computed by formula (1), where now $x_M = y_M = 0$:

$$d = \frac{|-1|}{\sqrt{(1/2)^2 + (1/3)^2}} = \frac{6}{\sqrt{13}}. \blacktriangle$$

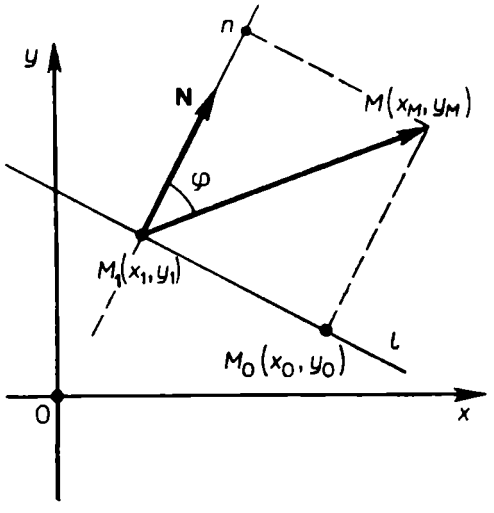

Fig. 32

If, along with the distance d, it is also required to determine the coordinates of the point $M_0(x_0, y_0)$, that is, the coordinates of the foot of the perpendicular dropped from the point $M(x_M, y_M)$ on the straight line l (see Fig. 32), then it is more convenient to take advantage of the following method of solving this problem. We draw through the point $M(x_M, y_M)$ a straight line l_1 perpendicular to the line l. Since the normal vector $\mathbf{N} = \{A, B\}$ of the line l is the direction vector for the line l_1, the canonical

equation of the line l_1 has the form:

$$\frac{x-x_M}{A}=\frac{y-y_M}{B}.$$

The point $M_0(x_0, y_0)$ belongs to the straight lines l and l_1. Consequently, its coordinates satisfy the system of equations

$$\begin{cases} Ax_0+By_0+C=0, \\ \dfrac{x_0-x_M}{A}=\dfrac{y_0-y_M}{B}. \end{cases}$$

The desired distance d is computed by the formula

$$d=\sqrt{(x_M-x_0)^2+(y_M-y_0)^2}.$$

Example 2. Given a triangle with vertices at the points A (1, 0), B (2, 3), and C (3, 1). Compute the length of the perpendicular BD dropped from the vertex B on the side AC and the length of the line segment AD.

△ Let us write the equation of the straight line AC passing through the points A (1, 0) and C (3, 1):

$$\frac{x-1}{3-1}=\frac{y-0}{1-0} \quad \text{or} \quad x-2y-1=0.$$

The normal vector of the straight line AC is represented by the vector $\mathbf{N}=\{1, -2\}$. Since this vector will be the direction vector for the straight line BD passing through the point B (2, 3) perpendicular to the straight line AC, the equation of the straight line BD has the form:

$$\frac{x-2}{1}=\frac{y-3}{-2} \quad \text{or} \quad 2x+y-7=0.$$

The coordinates x_0, y_0 of the point of intersection D of the straight lines AC and BD are found from the system of equations

$$\begin{cases} x_0-2y_0-1=0, \\ 2x_0+y_0-7=0, \end{cases}$$

whence $x_0=3$, $y_0=1$. We denote the length of the perpendicular BD by d and the length of the line segment AD by L. Then

$$d=\sqrt{(2-3)^2+(3-1)^2}=\sqrt{5} \text{ and } L=\sqrt{(3-1)^2+(1-0)^2}=\sqrt{5}. \blacktriangle$$

Exercises

1. Compute the angle between two straight lines:

(a) $\dfrac{x-2}{3}=\dfrac{y}{-3\sqrt{3}}$ and $\dfrac{x+5}{-3}=\dfrac{y-1}{-3\sqrt{3}}$;

(b) $\frac{x+5}{24}=\frac{y-4}{7}$ and $\frac{x-2}{-15}=\frac{y+\frac{7}{8}}{8}$;

(c) $x + 5y + 9 = 0$ and $2x - 3y + 1 = 0$;

(d) $2x + y - 5 = 0$ and $3x - y + 4 = 0$;

(e) $y = -2x + 7$ and $y = 3x + 4$.

2. Which of the following pairs of straight lines are pairs of parallel or mutually perpendicular lines:

(a) $\frac{x+1}{2}=\frac{y-4}{5}$ and $\frac{x-9}{4}=\frac{y+3}{10}$;

(b) $\frac{x-5}{2}=\frac{y}{3}$ and $\frac{x+1}{3}=\frac{y+9}{-2}$;

(c) $2x - 3y - 7 = 0$ and $4x - 6y + 5 = 0$;

(d) $2x + y - 3 = 0$ and $3x - y + 5 = 0$.

3. For what a are the straight lines $y = ax + 3$ and $y = -3x + 2$ parallel?

4. For what value of a are the straight lines $y = ax - 1$ and $y = 5x + 3$ mutually perpendicular?

5. Given a straight line $2x - 3y + 5 = 0$. Write the equation of the straight line passing through the point M (4, —5): (a) parallel to the given straight line; (b) perpendicular to the given line.

6. Given a straight line $\frac{x}{3}+\frac{y}{-2}=1$. Determine the distance from this line to the origin.

7. Find the distance from the given point to the given straight line:

(a) $M\left(\frac{3}{2}, 9\right)$, $4x+3y-8=0$; (b) $M\left(-\frac{3}{2}, -9\right)$, $4x + 3y-17=0$.

8. Compute the distance between the parallel lines: (a) $4x - 3y + 25 = 0$ and $8x - 6y + 25 = 0$; (b) $5x - 12y + 26 = 0$ and $5x - 12y - 13 = 0$.

9. Determine the coordinates of the point N which is symmetric to the point M (1, 2) with respect to the straight line passing through the points A (1, 0) and B (—1, —2).

10. Given a triangle with vertices at the points A (3, 6), B (—1, 3), and C (2, —1). Compute the length of its altitude emanating from the vertex C.

11. Given three vertices of a parallelogram: A (3, 7), B (2, —3), and C (—1, 4). Compute the length of its altitude dropped from the vertex B on the side AC.

12. Given two vertices of a triangle: A (—10, 2) and B (6, 4). Its altitudes intersect at the point M (5, 2). Find the third vertex C.

2.4. Second-order Curves (Quadric Curves)

1. The Equation of Second Degree in Two Variables. The Circle. In section 2.2, it was shown that if there is a Cartesian coordinate system introduced in the plane, then any equation of the first degree in two variables, that is, any equation of the form

$$Ax + By + C = 0,$$

where the coefficients A and B are not equal to zero simultaneously, is an equation of a straight line. And vice versa, any straight line is given by a first-degree equation in two variables. Therefore the straight line is sometimes referred to as a first-order line.

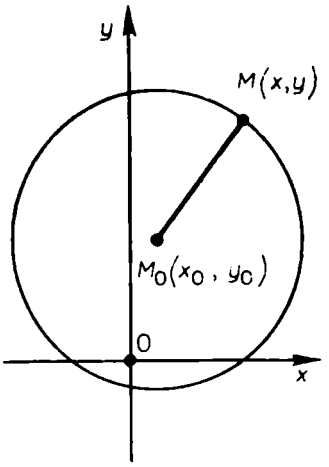

Fig. 33

Now we shall consider second-degree equations in two variables. The curves described by these equations are called second-order curves. The circle is the simplest second-order curve. Recall that the circle of radius R with centre at the point M_0 is defined as the set of all points M in the plane which meet the condition (Fig. 33)

$$M_0M = R. \qquad (1)$$

Let the point M_0 have coordinates x_0, y_0 in some Cartesian coordinate system Oxy. Then from condition (1) we get the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \qquad (2)$$

This second-degree equation in two variables is the equation of a circle of radius R with centre at the point M_0 (x_0, y_0).

Example 1. Write the equation of the circle of radius $R = 8$ with centre at the point M_0 $(2, -5)$.

△ Substituting the numerical values of the coordinates of the point M_0 and the radius into formula (2), we get $(x - 2)^2 + (y - (-5))^2 = 8^2$ or $(x - 2)^2 + (y + 5)^2 = 64$. ▲

Example 2. Prove that the equation $x^2 + y^2 + 8x - 4y - 5 = 0$ is the equation of a circle. Find its centre and radius.

△ We transform the left-hand side of the given equation: $x^2 + 8x + 16 - 16 + y^2 - 4y + 4 - 4 - 5 = 0$ or $(x + 4)^2 + (y - 2)^2 = 5^2$.

This equation represents the equation of the circle centred at the point M_0 $(-4, 2)$ with radius equal to 5. ▲

2. The Ellipse. The ellipse is defined as the set of points in the plane for each of which the sum of the distances from two given

points in the same plane is a constant and is greater than the distance between these points. The given points are called the foci of the ellipse, and the distance between them is termed the focal length. Let us deduce the equation of the ellipse. For this purpose, we arrange the coordinate system so that the axis of abscissas passes through the foci of the ellipse F_1 and F_2 (Fig. 34). We denote the focal distance, that is, the length of the line segment F_1F_2 by $2c$, and draw the axis of ordinates through the midpoint of F_1F_2 perpendicular to this segment. We now denote the sum of the lengths of the line segments F_1M and F_2M by $2a$ (this sum is constant for all points $M(x, y)$ of the ellipse). Then the coordinates of its points satisfy the equation

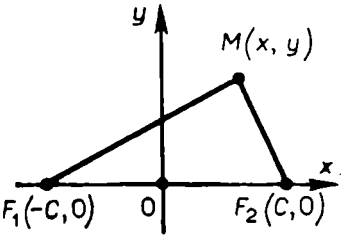

Fig. 34

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a. \qquad (1)$$

Isolating one of the radicals, we square both sides of the equation:

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2.$$

After simplifications we get:

$$\sqrt{(x-c)^2+y^2}=a-\frac{cx}{a}.$$

We square both sides of the equation and collect like terms:

$$\frac{a^2-c^2}{a^2}x^2+y^2=a^2-c^2.$$

By the definition of the ellipse, $a>c$, therefore (a^2-c^2) is a positive number. We denote it by b^2, that is, we set $b^2=a^2-c^2$. On being divided by b^2, the equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (2)$$

This equation is called the canonical equation of the ellipse. We can show that equation (2) is equivalent to equation (1).

Example 1. Write the canonical equation of the ellipse passing through the point $M(6, 0)$ if the focal distance is equal to 4.

△ We write the canonical equation of the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

By hypothesis, the point $M(6, 0)$ belongs to the ellipse, consequently, $\frac{36}{a^2}=1$, whence $a^2=36$. It remains to find the parameter

$b^2 = a^2 - c^2$, where c is half the focal distance of the ellipse. By hypothesis, $c = 2$. Then $b^2 = 36 - 4 = 32$. Thus, the desired equation of the ellipse is

$$\frac{x^2}{36} + \frac{y^2}{32} = 1. \blacktriangle$$

Example 2. Prove that the equation $64x^2 + 100y^2 - 6400 = 0$ is the equation of an ellipse. Find the coordinates of its foci and focal distance.

△ Dividing both sides of the equation by 6400, we get

$$\frac{x^2}{100} + \frac{y^2}{64} = 1.$$

This equation is the canonical equation of the ellipse. From the equality $a^2 - c^2 = b^2$ it follows that $c^2 = a^2 - b^2 = 100 - 64 = 36$,

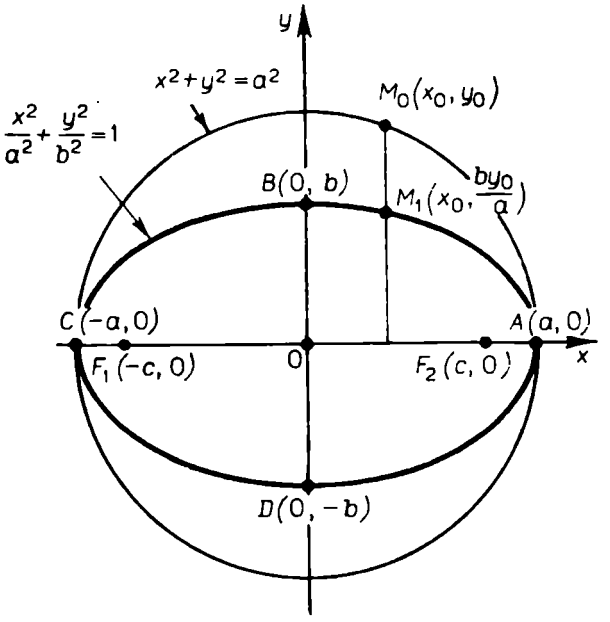

Fig. 35

and $c = 6$. The foci of the ellipse are found at the points F_1 $(-6, 0)$ and F_2 $(6, 0)$. The focal distance $2c = 12$. ▲

Let us now investigate the shape of the ellipse. We see from equation (2) that the ellipse is symmetric with respect to the coordinate axes (Oy and Ox) and the origin. Indeed, if some point M_0 (x_0, y_0) belongs to the ellipse, then the points M_1 $(-x_0, y_0)$, M_2 (x_0, y_0), M_3 $(-x_0, -y_0)$ also belong to it. The ellipse intersects the coordinate axes at the points A $(a, 0)$, C $(-a, 0)$ (these are the points of intersection with the x-axis), B $(0, b)$, and D $(0, -b)$ (these are the points of intersection with the y-axis). The points A, B, C, and D

are called the vertices of the ellipse. The line segment AC is called the major axis of the ellipse, and the line segment BD its minor axis. The foci F_1 and F_2 of the ellipse lie on its major axis. The length of the major axis is equal to $2a$, the length of the minor axis to $2b$. The numbers a and b are known as the semiaxes of the ellipse. The ellipse has two axes of symmetry which are mutually perpendicular. The point of intersection of the axes of symmetry of the ellipse is called its centre.

Ellipse (2) can be obtained from the circle whose equation is

$$x^2+y^2=a^2 \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \tag{3}$$

by compressing it towards the x-axis. This compression results in the ordinates of the points of the circle decreased $\left(\frac{b}{a}\right)$ times. Indeed, if the point $M_0(x_0, y_0)$ belongs to circle (3), that is, satisfies the equation $x_0^2+y_0^2=a^2$, then the coordinates of the point $M_1(x_0, y_1)$, where $y_1=\frac{by_0}{a}$, satisfy the equation of ellipse (2). The general shape of the ellipse and circle from which it can be obtained by compressing towards the x-axis is shown in Fig. 35.

3. The Hyperbola. The hyperbola is defined as the set of points in the plane for each of which the absolute values of the distances from two given points in the same plane have a constant difference and is less than the distance between these points. The given points are called the foci of the hyperbola, and the distance between them is the focal length. Let us write the equation of the hyperbola in coordinate form. To this end, a system of coordinates is chosen so that the axis of abscissas passes through the foci of the hyperbola F_1 and F_2 (see Fig. 36). The focal length, that is, the length of the line segment F_1F_2, will be denoted by $2c$. The axis of ordinates passes through the midpoint of the line segment F_1F_2 perpendicular to it. Let us denote the absolute value of the difference between the line segments F_1M and F_2M (which is constant for all the points of the hyperbola) by $2a$. Then the coordinates of the points of the hyperbola satisfy the equation

$$|\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}|=2a \text{ or}$$
$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a,$$

where the plus sign corresponds to the case when the left-hand side of the equation is positive, the minus sign corresponding to the opposite case. Isolating one of the radicals, we square both sides of the equation

$$(x+c)^2+y^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2.$$

After simplifications we get:

$$\frac{cx}{a} - a = \pm \sqrt{(x-c)^2 + y^2}.$$

We square both sides of the equation and collect like terms:

$$\frac{c^2 - a^2}{a^2} x^2 - y^2 = c^2 - a^2.$$

By the definition of the hyperbola, $a < c$ ($2a < 2c$, that is, the difference of the lengths of the sides F_1M and F_2M in the triangle

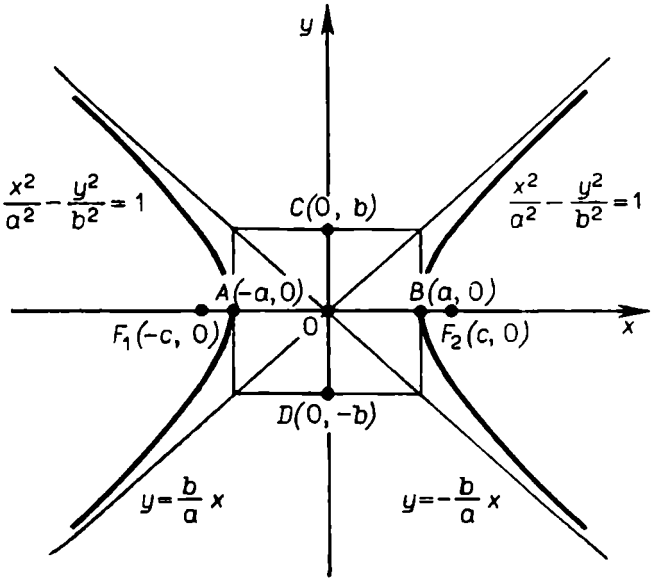

Fig. 36

F_1MF_2 (see Fig. 36) is less than the length of the side F_1F_2 in absolute value), therefore $(c^2 - a^2)$ is a positive number. We denote it by b^2, that is, we set $b^2 = c^2 - a^2$. On having been divided by b^2, the equation takes the form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (1)$$

Equation (1) is called the canonical equation of the hyperbola.

Example 1. Write the canonical equation of the hyperbola passing through the point M $(8\sqrt{5}, 12)$ if its focal distance is equal to 20.

△ By hypothesis, the focal distance $2c$ is equal to 20, that is, $c = 10$. We then write the canonical equation of the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

By hypothesis, the point $M(8\sqrt{5}, 12)$ belongs to the hyperbola, consequently,

$$\frac{320}{a^2} - \frac{144}{b^2} = 1.$$

This equation for determining a^2 and b^2 yields the relation

$$b^2 = c^2 - a^2 = 100 - a^2.$$

Solving the system of equations

$$\begin{cases} \dfrac{320}{a^2} - \dfrac{144}{b^2} = 1, \\ b^2 = 100 - a^2 \end{cases}$$

with respect to a^2 and b^2 (of course, only positive solutions are suitable), we find $a^2 = 64$, $b^2 = 36$. Thus, we get the desired equation $\dfrac{x^2}{64} - \dfrac{y^2}{36} = 1$. ▲

Example 2. Prove that the equation $21x^2 - 43y^2 = 903$ is the equation of a hyperbola. Find the coordinates of its foci.

△ Dividing both sides of the equation by 903, we get

$$\frac{x^2}{43} - \frac{y^2}{21} = 1.$$

This is the equation of the hyperbola for which $a^2 = 43$ and $b^2 = 21$. From the relation $c^2 = a^2 + b^2$ we find: $c^2 = 64$ and $c = 8$. Consequently, the foci of the hyperbola are situated at the points F_1 $(-8, 0)$, and F_2 $(8, 0)$. ▲

We now investigate the shape of the hyperbola. As is seen from equation (1), the hyperbola is symmetric with respect to the x- and y-axes and the origin O. Indeed, if the point $M_0(x_0, y_0)$ belongs to the hyperbola, then the points $M_1(-x_0, y_0)$, $M_2(x_0, -y_0)$, and $M_3(-x_0, -y_0)$ also belong to it. The hyperbola intersects the axis of abscissas at the points $A(-a, 0)$ and $B(a, 0)$. The points A and B are called the vertices of the hyperbola. The line segment AB is called the transverse axis of the hyperbola. Its length is $2a$. The hyperbola consists of two separate curves called its branches (Fig. 36). The straight lines

$$y = \frac{bx}{a} \text{ and } y = -\frac{bx}{a}$$

are called the asymptotes of the hyperbola: as x increases in absolute value, the branches of the hyperbola approach these straight lines still closer and closer (see Fig. 36). If $a = b$, then the hyperbola is termed rectangular (or equiangular or equilateral). Its equation is

$$x^2 - y^2 = a^2.$$

The asymptotes of a rectangular hyperbola are mutually perpendicular straight lines whose equations are $y = x$ and $y = -x$.

4. The Parabola. The parabola is the set of all points in the plane each of which is equidistant from a given point and a given straight line not passing through the given point. The given point is called the focus, and the given straight line the directrix of the parabola. The distance from the focus to the directrix is the focal parameter of the parabola and is denoted by p. In order to obtain the equation of the parabola in a simple form, a system of coordinates should be chosen in the following way: the x-axis is drawn through the focus F perpendicular to the directrix d. The point of intersection of the x-axis and directrix will be denoted by D (Fig. 37). For the coordinate origin we take the midpoint O of the line segment DF, and for the positive direction of the x-axis, the direction of the vector $\overrightarrow{OF}$. Then in the chosen coordinate system the focus F has the coordinates $\left(\frac{p}{2}, 0\right)$, while the directrix has the equation

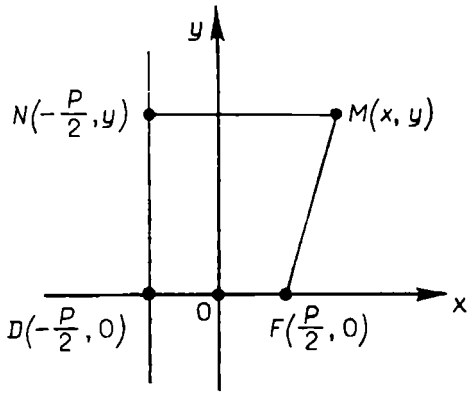

Fig. 37

$$x = -\frac{p}{2} \quad \text{or} \quad x + \frac{p}{2} = 0.$$

Let $M(x, y)$ be an arbitrary point of the parabola. From this point we drop a perpendicular onto the directrix d, and let N be the foot of the perpendicular. The length of the line segment MN is equal to the distance from the point M to the directrix. The distance from the point M to the focus is equal to the length of the line segment MF. Then the coordinates of the points of the parabola satisfy the equation

$$\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \left| x + \frac{p}{2} \right|.$$

Squaring both sides of the equation and collecting like terms we get:

$$y^2 = 2px. \tag{1}$$

Equation (1) is the canonical equation of the parabola.

Example. Given a parabola: $y^2 = 3x$. Find the points of the given parabola whose distance from the focus is equal to 1.

△ Since $2p=3$, we have $\frac{p}{2}=\frac{3}{4}$, and the focus of the parabola is found at the point $F\left(\frac{3}{4}, 0\right)$. Let $M(x, y)$ be a sought-for point belonging to the parabola. Since, by hypothesis, the distance from this point to the focus is equal to unity, then, to find the coordinates x and y, we have to solve the system of equations

$$\begin{cases} \left(x-\frac{3}{4}\right)^2+y^2=1, \\ y^2=3x. \end{cases}$$

Solving this system of equations, we get:

$$\left(x+\frac{3}{4}\right)^2=1, \quad x+\frac{3}{4}=1, \quad x=\frac{1}{4}.$$

Then $y^2=3x=3\cdot\frac{1}{4}=\frac{3}{4}$ and $y=\pm\frac{\sqrt{3}}{2}$. Thus, there are two points whose distance to the focus is equal to 1: $\left(\frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right)$. ▲

Let us investigate the shape of the parabola. As is seen from equation (1), the parabola lies entirely in the domain $x\geqslant 0$ and is symmetric with respect to the x-axis. Indeed, if the point $M_0(x_0, y_0)$ belongs to the parabola, then the point $M_1(x_0, -y_0)$ also belongs to it (Fig. 38). The axis of symmetry of the parabola is called the axis of the parabola. Parabola (1) intersects the axis of symmetry Ox at the point $O(0, 0)$ which is called the vertex of the parabola. The equation

$$x^2=2py \quad \text{or} \quad y=\frac{1}{2p}\cdot x^2$$

corresponds to the parabola which has the same shape as parabola (1) but is arranged in a different way: its axis of symmetry coincides with the y-axis.

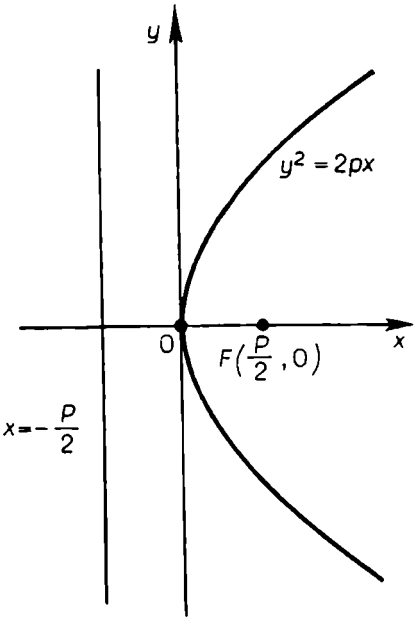

Fig. 38

5. The General Equation of the Second Degree. In a Cartesian coordinate system, the general second-degree equation has the form:

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \qquad (1)$$

where $A^2+B^2+C^2\neq 0$. In the theory of second-order curves which is beyond the scope of the present course of higher mathem-

atics it is proved that it is always possible to change to a new coordinate system in which equation (1) will have one of the following nine forms

$$(1)\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \quad (2)\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1; \quad (3)\ y^2=2px;$$

$$(4)\ a^2x^2-b^2y^2=0; \quad (5)\ y^2-a^2=0; \quad (6)\ y^2=0;$$

$$(7)\ a^2x^2+b^2y^2=0; \quad (8)\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=-1; \quad (9)\ y^2+a^2=0,$$

where the coefficients a, b, and p are not equal to zero. Equations 1-9 are called canonical. Equations 8 and 9 correspond to empty sets since they have no real solutions. Equation 7 describes the only point in the plane O (0, 0). The sets of points in the plane defined by equations 4-6 consist of straight lines. Equation 6 defines the only straight line $y = 0$; equation 5 two straight lines $y - a = 0$ and $y + a = 0$; equation 4 also describes two straight lines: $ax - by = 0$ and $ax + by = 0$. Equations 1-3 correspond to the new geometrical objects: the ellipse, the hyperbola, and the parabola studied in the present section. The circle is a particular case of the ellipse when $a = b$ in equation 1.

In practice, second-order curves are encountered when studying the trajectories of celestial bodies: the planets are moving around the Sun in elliptical orbits, all the comets of the solar system move either in an ellipse or in a parabola or in a hyperbola.

Exercises

1. Find the centre and radius of the given circle:

(a) $x^2 + y^2 = 21$;

(b) $(x - 2)^2 + (y - 4)^2 = 36$;

(c) $(x+5)^2+\left(y+\frac{3}{4}\right)^2=17$.

2. Prove that the given equation is the equation of a circle. Find its centre and radius:

(a) $x^2 - 2x + 4y + y^2 - 20 = 0$;

(b) $x^2 - 6x + 10y - y^2 + 9 = 0$.

3. Write the equation of the circle centred at the point C (2, 5) if it is known that the circle touches the x-axis.

4. Write the equation of the circle centred at the point B (2, —1) and passing through the point A (6, 2).

5. Determine the position of the point $M(-2, 1)$ relative to each of the given circles (lies inside, outside, or on the circle):

(a) $x^2 + y^2 = 2$; (b) $x^2 + y^2 - 5 = 0$;

(c) $x^2 + y^2 = 25$; (d) $x^2 + y^2 - 8x - 4y = 5$;

(e) $x^2 + y^2 + 6x - 8y = 0$; (f) $x^2 + y^2 = 0.01$.

6. Determine the position of the straight line relative to the circle (intersects, touches, that is, has one point in common, or passes outside it) if the straight line and circle are given by the following equations:

(a) $2x - y - 3 = 0$ and $x^2 + y^2 - 3x + 2y - 3 = 0$;

(b) $x - 2y - 1 = 0$ and $(x - 4)^2 + (y + 1)^2 = 5$;

(c) $x + 3y + 10 = 0$ and $x^2 + y^2 = 1$.

7. Write the equation of the circle passing through the following three points: $M_1(0, 0)$, $M_2(3, 0)$, and $M_3(0, 4)$.

8. Write the equation of the circle circumscribed about the triangle whose sides belong to the straight lines $x - 3y + 1 = 0$, $9x - 2y - 41 = 0$, and $7x + 4y + 7 = 0$.

9. Write the canonical equation of the ellipse if its focal distance is equal to 8, and the ellipse passes through the point $(0, -3)$.

10. Write the canonical equation of the ellipse if its focus is found at the point $(6, 0)$, and the axis of ordinates intersects the ellipse at the point $(0, -3)$.

11. Write the canonical equation of the ellipse if: (a) its semiaxes are equal to 5 and 3; (b) its semimajor axis is equal to 6, and the focal distance is 7.

12. For each of the following ellipses, determine its semiaxes and the coordinates of its vertices and foci:

(a) $9x^2 + 16y^2 = 144$; (b) $x^2 + 9y^2 = 4$; (c) $4x^2 + 9y^2 = 1$.

13. The ordinates of the points of the circle $x^2 + y^2 = 25$ are reduced by half in absolute value. Write the equation of the newly obtained curve.

14. Given an ellipse: $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Find its semimajor and semiminor axes, focal distance, and coordinates of the foci and vertices.

15. Write the canonical equation of the ellipse in which the distances from the focus to the endpoints of the major axis are equal to 1 and 9.

16. Compute the area of the quadrilateral two vertices of which are found at the foci of the ellipse $9x^2 + 25y^2 - 225 = 0$, and two other vertices coincide with the endpoints of its minor axis.

17. Write the canonical equation of the hyperbola if its focal distance is equal to 30, and the hyperbola passes through the point $(-9, 0)$.

18. Determine the semiaxes of each of the following hyperbolas:

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$; (b) $16x^2 - y^2 = 1$;

(c) $x^2 - 9y^2 = 9$; (d) $x^2 - y^2 = 4$.

19. For the hyperbola $9x^2 - 16y^2 - 144 = 0$, find: (a) the semiaxes; (b) the coordinates of the foci; (c) the coordinates of the vertices; (d) the equations of the asymptotes.

20. Write the canonical equation of the hyperbola if the distances from one of its vertices to the foci are equal to 9 and 1.

21. Write the canonical equation of the hyperbola if the equation of its asymptote is $y = \frac{1}{2}x$, and one of its foci is found at the point $(-5, 0)$.

22. Write the equation of the hyperbola whose foci are located at the vertices of the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$, and the vertices at the foci of the ellipse.

23. Write the equation of the parabola if the coordinates of its focus are $(4, 0)$, and the equation of the directrix is $x + 4 = 0$.

24. Write the canonical equation of the parabola passing through the point $(5, 3)$.

25. Given a parabola: $y^2 = 5x$. Find the points of the parabola located at a distance of 4 from the focus.

26. Find the points of intersection of the parabola $y^2 = 4x$ and the straight lines: (a) $x = y$; (b) $x = -y$; (c) $x - 2y + 4 = 0$; (d) $3x - 2y + 1 = 0$.

27. Write the equation of the parabola passing through the origin and symmetric with respect to the axis of ordinates if the coordinates of the focus are $F(0, -3)$.

Chapter 3

ANALYTIC GEOMETRY IN SPACE

3.1. The Plane

1. The Equation of a Plane Passing Through Three Given Points. As is known, a unique plane passes through three noncollinear points in space. Let us find the equation of the plane p passing through three points M_1, M_2, and M_3 if their Cartesian coordinates are known: $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, and $M_3(x_3, y_3, z_3)$.

A point $M(x, y, z)$ belongs to the plane p if and only if the vectors $\overrightarrow{M_1M} = \{x - x_1, y - y_1, z - z_1\}$, $\overrightarrow{M_1M_2} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$, $\overrightarrow{M_1M_3} = \{x_3 - x_1, y_3 - y_1, z_3 - z_1\}$ are coplanar, that is, if their scalar triple product is equal to zero

$$(\overrightarrow{M_1M}, \overrightarrow{M_1M_2}, \overrightarrow{M_1M_3}) = 0. \tag{1}$$

Since the scalar triple product of three vectors is equal to the third-order determinant formed from the coordinates of these vectors, the sought-for equation of the plane passing through the three given points has the form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0. \tag{2}$$

Example. Write the equation of the plane passing through the points $M_1(1, 2, 3)$, $M_2(-1, 2, 1)$, and $M_3(3, -1, 1)$.

△ In the present case, $x_1 = 1$, $y_1 = 2$, $z_1 = 3$, $x_2 = -1$, $y_2 = 2$, $z_2 = 1$, $x_3 = 3$, $y_3 = -1$, and $z_3 = 1$.

Substituting these values into equation (2), we have

$$\begin{vmatrix} (x-1) & (y-2) & (z-3) \\ -2 & 0 & -2 \\ 2 & -3 & -2 \end{vmatrix} = 0.$$

We expand the third-order determinant in terms of the elements of the first row:

$$(x-1)\begin{vmatrix} 0 & -2 \\ -3 & -2 \end{vmatrix} - (y-2)\begin{vmatrix} -2 & -2 \\ 2 & -2 \end{vmatrix} + (z-3)\begin{vmatrix} -2 & 0 \\ 2 & -3 \end{vmatrix} = 0.$$

Evaluating the second-order determinants, we find the required equation:

$$-6(x-1)-8(y-2)+6(z-3)=0$$

or

$$3x+4y-3z-2=0. \blacktriangle$$

2. The Equation of a Plane Passing Through a Given Point Perpendicular to a Given Vector. A plane in space is uniquely defined by a point through which it passes and a normal (orthogonal) vector. Let us find the equation of the plane p passing through the point M_1 perpendicular to the vector $\mathbf{N} \neq \mathbf{0}$ if their coordinates are known:

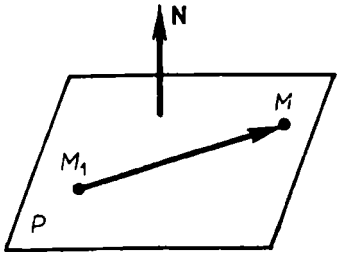

Fig. 39

$$M_1(x_1, y_1, z_1), \quad \mathbf{N}=\{A, B, C\}.$$

The point $M(x, y, z)$ belongs to the plane p if and only if the vector $\overrightarrow{M_1M}=\{x-x_1, y-y_1, z-z_1\}$ is perpendicular to the vector $\mathbf{N}$ (Fig. 39), that is, when the following equality holds true:

$$\mathbf{N}\cdot\overrightarrow{M_1M}=0. \qquad (1)$$

Since the scalar product of two vectors is equal to the sum of products of like coordinates of these vectors, equation (1) takes the form:

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0. \qquad (2)$$

Equation (2) is the equation of the plane p passing through the point $M_1(x_1, y_1, z_1)$ perpendicular to the given vector $\mathbf{N}=\{A, B, C\}$.

Example 1. Find the equation of the plane passing through the point $M(-1, 9, 5)$ perpendicular to the vector $\mathbf{N}=\{1, -1, 4\}$.

$\triangle$ Here, $x_1=-1$, $y_1=9$, $z_1=5$, and $A=1$, $B=-1$, $C=4$. Substituting these values into equation (2), we get the desired equation

$$1\cdot(x+1)-1\cdot(y-9)+4\cdot(z-5)=0$$

or

$$x-y+4z-10=0. \blacktriangle$$

Example 2. Given two points $M_1(4, 5, 0)$ and $M_2(2, -1, 3)$. Write the equation of the plane passing through the point M_1 perpendicular to the vector $\overrightarrow{M_2M_1}$.

$\triangle$ For the normal vector of the plane we take the vector $\mathbf{N}=\overrightarrow{M_2M_1}=\{4-2, 5-(-1), 0-3\}=\{2, 6, -3\}$. Substituting the coordinates of the normal vector and point $M_1(4, 5, 0)$ into equation (2), we get

$$2\cdot(x-4)+6\cdot(y-5)-3\cdot(z-0)=0 \text{ or}$$

$$2x+6y-3z-38=0. \blacktriangle$$

Denoting the number $-(Ax_1 + By_1 + Cz_1)$ by D, we write the equation of plane (2) in the form

$$Ax + By + Cz + D = 0, \tag{3}$$

where $A^2 + B^2 + C^2 \neq 0$.

Thus, any plane in space is defined by a linear equation in three variables. This equation is called the general equation of the plane. We emphasize here that in this equation the coefficients A, B, and C are the coordinates of the normal vector of the plane. For instance, if a plane is given by the equation $5x + 2y - z + 11 = 0$, then we can say at once that it is perpendicular to the vector $\mathbf{N} = \{5, 2, -1\}$.

Example 3. Find the unit normal vector of the plane

$$\sqrt{41}x + 2y - 2z + 3 = 0.$$

△ As a normal vector of the given plane, we may take the vector $\mathbf{N} = \{\sqrt{41}, 2, -2\}$ whose length is equal to $|\mathbf{N}| = \sqrt{41 + 4 + 4} = 7$. Consequently as the unit normal vector we have the vector $\mathbf{n} = \left\{\frac{\sqrt{41}}{7}, \frac{2}{7}, -\frac{2}{7}\right\}$, and its opposite vector

$$-\mathbf{n} = \left\{-\frac{\sqrt{41}}{7}, -\frac{2}{7}, \frac{2}{7}\right\}. \blacktriangle$$

Example 4. Write the equation of the plane which passes through the point M_1 (1, 0, 0) parallel to the plane $x + y + z = 0$.

△ The normal vector of the plane $x + y + z = 0$ is represented by the vector $\mathbf{N} = \{1, 1, 1\}$. The same vector will be the normal vector for the required plane since the planes are parallel to each other. We write the equation of the plane passing through the point M_1 (1, 0, 0) perpendicular to the vector $\mathbf{N} = \{1, 1, 1\}$:

$$1 \cdot (x - 1) + 1 \cdot (y - 0) + 1 \cdot (z - 0) = 0$$

or

$$x + y + z - 1 = 0. \blacktriangle$$

Exercises

1. Write the equation of the plane passing through the points M_1, M_2, and M_3 if: (a) M_1 (−2, 3, 5), M_2 (4, −3, 0), M_3 (0, 6, −5); (b) M_1 (2, 0, 4), M_2 (3, 1, −2), M_3 (0, −3, −1); (c) M_1 (3, 1, −5), M_2 (8, 3, 3), M_3 (−2, −1, 4).

2. Write the equation of the plane passing through the point M_1 perpendicular to the vector $\mathbf{N}$ if: (a) M_1 (2, 3, 5), $\mathbf{N} = \{4, 6, 0\}$; (b) M_1 (3, −5, −2), $\mathbf{N} = \{4, -6, 1\}$; (c) M_1 (0, 0, 0), $\mathbf{N} = \{0, -7, 4\}$; (d) M_1 (1, 2, 3), $\mathbf{N} = \{0, 1, 0\}$.

3. Write the equation of the plane which passes through the point M_1 parallel to the given plane if: (a) $M_1(1, -5, 4)$, $4x - 7z + 6 = 0$; (b) $M_1(3, 4, -11)$, $z = 0$; (c) $M_1(2, -1, 3)$, $\frac{x}{2} - \frac{y}{7} + \frac{z}{2} = 1$.

4. Find the equation of the plane which passes through the point M_1 (1, 2, 5) and the axis of abscissas.

3.2. The Straight Line in Space

1. The Equation of a Straight Line Through Two Points. The same as in the plane, in space a unique straight line passes through two given points. Let us find the equation of the straight line l passing through two given points M_1 and M_2 if their coordinates are known: $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$.

A point $M(x, y, z)$ lies on the straight line l if and only if the vectors $\overrightarrow{M_1M} = \{x - x_1, y - y_1, z - z_1\}$ and $\overrightarrow{M_2M_1} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$ are collinear, that is, if the equality $\overrightarrow{M_1M} = \lambda\overrightarrow{M_2M_1}$ is valid, where λ is some number (see 1.1). The vector equality implies the equalities of like coordinates: $x - x_1 = \lambda(x_2 - x_1)$, $y - y_1 = \lambda(y_2 - y_1)$, $z - z_1 = \lambda(z_2 - z_1)$. Hence we find the equations of the straight line in space passing through two given points:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}, \quad \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \quad \text{or} \quad \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}. \tag{1}$$

Example. Write the equations of the straight line passing through the points $M_1(-2, 1, 5)$ and $M_2(4, 3, -1)$.

△ In the present case, $x_1 = -2$, $y_1 = 1$, $z_1 = 5$, $x_2 = 4$, $y_2 = 3$, $z_2 = -1$. Substituting these values into equations (1), we get:

$$\frac{x+2}{6} = \frac{y-1}{2} = \frac{z-5}{-6}. \quad ▲$$

2. The Equation of a Straight Line Passing Through a Given Point Parallel to a Given Vector. Let a point $M_1(x_1, y_1, z_1)$ and a vector $\mathbf{t} = \{a, b, c\}$ be given in a Cartesian coordinate system. Let us find the equation of the straight line l passing through the given point M_1 parallel to the given vector $\mathbf{t}$. The vector $\mathbf{t}$ is called the direction vector of the straight line l.

A point $M(x, y, z)$ lies on the straight line l if and only if the vectors $\overrightarrow{M_1M} = \{x - x_1, y - y_1, z - z_1\}$ and $\mathbf{t}$ are collinear, that

is, if the following equality is valid:

$$\overrightarrow{M_1M} = \lambda t, \tag{1}$$

where λ is a certain number (see 1.1). From vector equality (1) there follow the equalities of like coordinates: $x - x_1 = \lambda a$, $y - y_1 = \lambda b$, $z - z_1 = \lambda c$. Hence we find the equations of the straight line l:

$$\frac{x-x_1}{a} = \frac{y-y_1}{b}, \quad \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{or} \quad \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}. \tag{2}$$

Equations (2) are called the canonical equations of the straight line in space.

Example 1. Write the canonical equations of the straight line passing through the point M_1 (—2, 2, 5) parallel to the vector $\mathbf{t} = \{3, 2, 1\}$.

△ In the present case, $x_1 = -2$, $y_1 = 2$, $z_1 = 5$, and $a = 3$, $b = 2$, $c = 1$. Then the canonical equations of straight line (2) will be written in the form $\frac{x+2}{3} = \frac{y-2}{2} = \frac{z-5}{1}$. ▲

The equations of straight line (2) are valid whenever $a \neq 0$, $b \neq 0$, and $c \neq 0$. Let now, for instance, $a = 0$, that is, the direction vector of the straight line is represented by the vector $\mathbf{t} = \{0, b, c\}$. In this case, from vector equality (1) we find that $x - x_1 = \lambda \cdot 0$, $y - y_1 = \lambda b$, and $z - z_1 = \lambda c$. Then the equations of the straight line in space will take the form

$$x = x_1, \quad \frac{y-y_1}{b} = \frac{z-z_1}{c}. \tag{3}$$

In the case $b = 0$ the following will be the equations of the straight line

$$y = y_1, \quad \frac{x-x_1}{a} = \frac{z-z_1}{c}, \tag{4}$$

and so on. For uniformity, equations (3) and (4) are customarily written as follows:

$$\frac{x-x_1}{0} = \frac{y-y_1}{b} = \frac{z-z_1}{c}; \quad \frac{x-x_1}{a} = \frac{y-y_1}{0} = \frac{z-z_1}{c}, \tag{5}$$

assuming that if the denominator is equal to zero, then the appropriate numerator is also zero.

Example 2. Write the canonical equations of the straight line passing through the point M_1 (—1, 3, —2) parallel to the vector $\mathbf{t} = \{1, 0, 1\}$.

△ Since here $a = 1$, $b = 0$, $c = 1$, we make use of the second of equations (5):

$$\frac{x+1}{1} = \frac{y-3}{0} = \frac{z+2}{1} \quad \text{or} \quad y = 3, \; x+1 = z+2. \; ▲$$

3. The Parallelism and Perpendicularity Conditions of a Straight Line and a Plane. The straight line with the direction vector **t** and the plane with normal vector **N** are parallel if and only if the vectors **t** and **N** are perpendicular (Fig. 40,*a*).

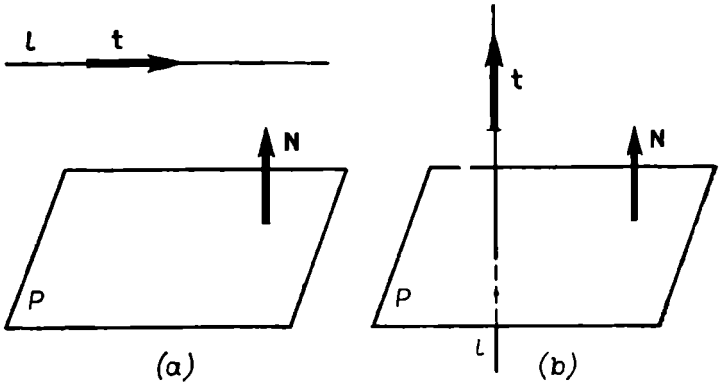

Fig. 40

It is obvious that for a straight line and a plane to be mutually perpendicular, it is necessary and sufficient that the vectors **t** and **N** be collinear (Fig. 40, *b*).

A straight line lies in a plane if and only if it is, firstly, parallel to the plane and, secondly, at least one of its points belongs to the plane.

Example 1. Indicate which of the given pairs of straight lines and planes are parallel or perpendicular; if a straight line pierces a plane, then find the point of their intersection:

$$\text{(a)}\ \frac{x-1}{3}=\frac{y+2}{3}=\frac{z}{-5}\quad\text{and}\quad 7x-2y+3z-1=0;$$

$$\text{(b)}\ \frac{x}{2}=\frac{y-1}{3}=\frac{z-1}{4}\quad\text{and}\quad x-y+z-3=0.$$

△ In (a), the direction vector of the straight line is represented by the vector $\mathbf{t}=\{3,\ 3,\ -5\}$, and the normal vector of the plane by the vector $\mathbf{N}=\{7,\ -2,\ 3\}$. These vectors are, obviously, noncollinear. Since $\mathbf{tN}=3\cdot 7-3\cdot 2-5\cdot 3=0$, the vector **t** is perpendicular to the vector **N**, that is, the given straight line and plane are parallel.

In (b), the vectors $\mathbf{t}=\{2,\ 3,\ 4\}$ and $\mathbf{N}=\{1,\ -1,\ 1\}$ are also noncollinear. However, we now have: $\mathbf{tN}=2\cdot 1-3\cdot 1+4\cdot 1\neq 0$, that is, the straight line and plane are not parallel and, consequently, they intersect. To determine the coordinates of the point of intersection $M_0\ (x_0,\ y_0,\ z_0)$, we have to solve the system of three equa-

tions in three unknowns:

$$\begin{cases} \frac{x_0}{2} = \frac{z_0 - 1}{4}, \\ \frac{y_0 - 1}{3} = \frac{z_0 - 1}{4}, \\ x_0 - y_0 + z_0 - 3 = 0. \end{cases}$$

Substituting the expressions $x_0 = \frac{z_0}{2} - \frac{1}{2}$ and $y_0 = \frac{3z_0}{4} + \frac{1}{4}$ found from the first two equations into the third equation, we get $\frac{3z_0}{4} - \frac{3}{4} - 3 = 0$, that is, $z_0 = 5$. Then $x_0 = 2$, and $y_0 = 4$. The straight line and plane intersect at the point $M_0(2, 4, 5)$. ▲

Example 2. Find the equation of the plane passing through the point $M_1(4, -3, 4)$ parallel to the straight lines $\frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$ and $\frac{x+1}{5} = \frac{y-3}{4} = \frac{z-4}{2}$.

△ The direction vectors of the given straight lines are $\mathbf{t}_1 = \{6, 2, -3\}$ and $\mathbf{t}_2 = \{5, 4, 2\}$. The sought-for plane is parallel to the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Consequently, as the normal vector $\mathbf{N}$ of this plane, we may take the vector product of the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$:

$$\mathbf{N} = \mathbf{t}_1 \times \mathbf{t}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -3 \\ 5 & 4 & 2 \end{vmatrix} = 16\mathbf{i} - 27\mathbf{j} + 14\mathbf{k}.$$

The equation of the plane passing through the given point $M_1(x_1, y_1, z_1)$ perpendicular to the given vector $\mathbf{N} = \{A, B, C\}$ has the form (see 3.1, subsection 2): $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$. Thus, the desired equation of the plane is

$$16(x - 4) - 27(y + 3) + 14(z - 1) = 0 \text{ or}$$
$$16x - 27y + 14z - 159 = 0. \blacktriangle$$

Example 3. Find the canonical equations of the straight line passing through the point $M_0(-8, 1, 5)$ perpendicular to the plane $2x - 3y + 5z - 4 = 0$.

△ Since the sought-for straight line is perpendicular to the given plane, for the direction vector of this straight line we may take the normal vector of the plane $\mathbf{N} = \{2, -3, 5\}$. Thereby we find the canonical equations of the desired straight line:

$$\frac{x+8}{2} = \frac{y-1}{-3} = \frac{z-5}{5}. \blacktriangle$$

Exercises

1. Write the canonical equations of the straight line passing through the point M_1 and having the direction vector $\mathbf{t}$ if: (a) M_1 $(-2, 0, 1)$ and $\mathbf{t} = \{2, -3, 4\}$; (b) M_1 $(2, -1, 0)$ and $\mathbf{t} = \{\sqrt{3}, \sqrt{2}, 1\}$; (c) M_1 $(3, 0, -3)$ and $\mathbf{t} = \{0, 1, 0\}$.

2. Write the canonical equations of the straight line passing through the points M_1 and M_2 if: (a) M_1 $(3, -1, 0)$ and M_2 $(-2, -5, 4)$; (b) M_1 $(1, -1, 4)$ and M_2 $(4, -1, 2)$; (c) M_1 $(0, 1, -5)$ and M_2 $(-2, 1, -5)$.

3. Investigate the mutual positions of the given straight lines and planes (if a straight line pierces a plane, then find the point of intersection):

(a) $\dfrac{x+7}{3} = \dfrac{y+10}{-4} = \dfrac{z+3}{1}$ and $y + 4z + 1 = 0$,

(b) $\begin{cases} 3x - 2y + z + 3 = 0, \\ 4x - 3y + 4z + 1 = 0 \end{cases}$ and $2x - y - 2z - 8 = 0$,

(c) $\dfrac{x-2}{-1} = \dfrac{y+1}{4} = \dfrac{z+5}{2}$ and $4x + y - 7 = 0$.

4. Find the equation of the straight line passing through the point M_1 $(0, 0, -1)$ perpendicular to the plane $2x + 3y - z + 7 = 0$.

5. Find the equation of the plane passing through the point $M_1(7, 9, 11)$ perpendicular to the straight line $\dfrac{x}{11} = \dfrac{y}{9} = \dfrac{z}{7}$.

3.3. Solving Certain Stereometric Problems

1. Solving Stereometric Problems by the Methods of Vector Algebra. Using the methods of vector algebra, we can solve various problems on computing angles, areas of faces and volumes of polyhedra, etc. Consider a couple of examples.

Example 1. Compute the volume of the pyramid whose vertices are located at the points A $(1, 2, -1)$, B $(0, 1, 5)$, C $(-1, 2, 1)$, D $(1, 2, 3)$.

△ The volume of the pyramid $ABCD$ is one-sixth of the volume of the parallelepiped with edges AB, AC, and AD. Therefore $V = \frac{1}{6} \cdot |(\vec{AB}, \vec{AC}, \vec{AD})|$. Let us evaluate the scalar triple product of three vectors $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ by formula (4) from 1.5, subsection 4:

$$(\vec{AB}, \vec{AC}, \vec{AD}) = \begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix} + 6\begin{vmatrix} -2 & 0 \\ 0 & 0 \end{vmatrix} = -8,$$

since $\overrightarrow{AB}=\{-1, -1, 6\}$, $\overrightarrow{AC}=\{-2, 0, 2\}$ and $\overrightarrow{AD}=\{0, 0, 4\}$. Consequently, $V=\frac{1}{6}\left|-8\right|=\frac{4}{3}$. ▲

Example 2. The length of the edge of the cube $OABCO_1A_1B_1C_1$ is equal to a (Fig. 41). The point K lies on the edge B_1B at a distance $\frac{a}{7}$ from the vertex B_1. Find the area of the section of the cube by the plane passing through the point K perpendicular to the straight line OK.

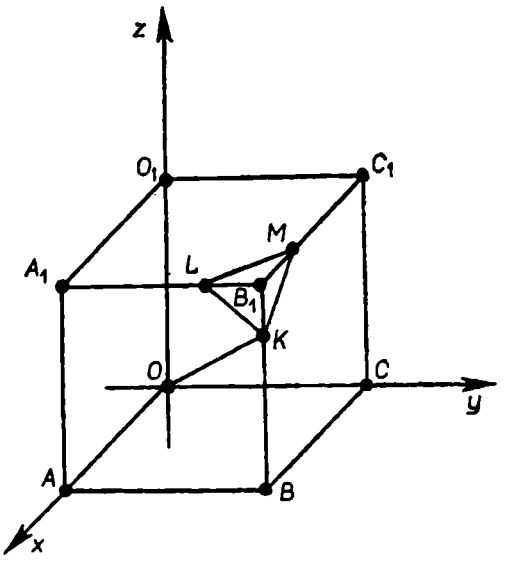

Fig. 41

△ We arrange the coordinate system $Oxyz$ in the way shown in Fig. 41. The cutting plane p passes through the point $K\left(a, a, \frac{6a}{7}\right)$ perpendicular to the vector $\overrightarrow{OK}=\left\{a, a, \frac{6a}{7}\right\}$. Let us write its equation:

$$a(x-a)+a(y-a)+\frac{6a}{7}\left(z-\frac{6a}{7}\right)=0. \qquad (1)$$

We denote the points of intersection of plane (1) and the edges B_1C_1 and A_1B_1 by $M(x_M, a, a)$ and $L(a, y_L, a)$, respectively. The coordinates of these points must satisfy equation (1). Substituting the coordinates of the points M and L into equation (1), we find that

$$a(x_M-a)+\frac{6a^2}{49}=0 \quad \text{and} \quad a(y_L-a)+\frac{6a^2}{49}=0,$$

whence $x_M=y_L=\frac{48a}{49}$. The area S_{Δ} of the triangle LMK obtained in the section of the cube by the plane p is found by the standard method: $S=\frac{1}{2}\left|\overrightarrow{KM}\times\overrightarrow{KL}\right|$. We compute the vector product of the vectors $\overrightarrow{KM}=\left\{-\frac{a}{49}, 0, \frac{a}{7}\right\}$ and $\overrightarrow{KL}=\left\{0, -\frac{a}{49}, \frac{a}{7}\right\}$:

$$\overrightarrow{KM}\times\overrightarrow{KL}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{49} & 0 & \frac{a}{7} \\ 0 & -\frac{a}{49} & \frac{a}{7} \end{vmatrix}=\frac{a^2}{7^3}\mathbf{i}+\frac{a^2}{7^3}\mathbf{j}+\frac{a^2}{7^4}\mathbf{k}.$$

Finally, we find that $S=\frac{1}{2}\sqrt{\frac{a^4}{7^6}+\frac{a^4}{7^6}+\frac{a^4}{7^8}}=\frac{3a^2\sqrt{11}}{4802}$. ▲

2. Computing the Distance from a Given Point to a Given Plane. When solving stereometrical problems, we often have to compute the distance from a point $M(x_M, y_M, z_M)$ given by its coordinates to some plane p defined by the general equation $Ax + By + Cz + D = 0$. To this end, we draw a straight line l through the point $M(x_M, y_M, z_M)$ perpendicular to the plane p. As the direction vector $\mathbf{t}$ of the straight line l, we can take the normal vector of the plane p, that is, the vector $\mathbf{N} = \{A, B, C\}$. Thus, $\mathbf{t} = \mathbf{N} = \{A, B, C\}$. We then write the canonical equation of the straight line:

$$\frac{x-x_M}{A} = \frac{y-y_M}{B} = \frac{z-z_M}{C}. \tag{1}$$

The coordinates of the point of intersection $M_0(x_0, y_0, z_0)$ of the straight line l and the plane p are found from the system of equations:

$$\begin{cases} Ax_0 + By_0 + Cz_0 + D = 0, \\ \dfrac{x_0 - x_M}{A} = \dfrac{z_0 - z_M}{C}, \\ \dfrac{y_0 - y_M}{B} = \dfrac{z_0 - z_M}{C}. \end{cases} \tag{2}$$

The desired distance d from the point M to the plane p is equal to the distance between the points M and M_0:

$$d = \sqrt{(x_M - x_0)^2 + (y_M - y_0)^2 + (z_M - z_0)^2}. \tag{3}$$

Example 1. Compute the distance from the origin to the plane $4x + \sqrt{11}y + 3z + 150 = 0$.

△ In the present case $x_M = y_M = z_M = 0$, $A = 4$, $B = \sqrt{11}$, $C = 3$, and $D = 150$. System of equations (2) takes the form:

$$\begin{cases} 4x_0 + \sqrt{11}\, y_0 + 3z_0 = -150, \\ \dfrac{x_0}{4} = \dfrac{z_0}{3}, \\ \dfrac{y_0}{\sqrt{11}} = \dfrac{z_0}{3}. \end{cases}$$

Substituting into the first equation the relations $x_0 = \frac{4}{3} z_0$, $y_0 = \frac{\sqrt{11}}{3} z_0$, we find $\left(\frac{16}{3} + \frac{11}{3} + 3\right) z_0 = -150$ or $z_0 = -\frac{25}{2}$. Then $x_0 = -\frac{50}{3}$, $y_0 = -\frac{25\sqrt{11}}{6}$. Using formula (3), we compute the required distance d:

$$d = \sqrt{(0 - x_0)^2 + (0 - y_0)^2 + (0 - z_0)^2}$$
$$= \sqrt{\frac{4 \cdot 25^2}{9} + \frac{11 \cdot 25^2}{36} + \frac{25^2}{4}} = 25. \ ▲$$

The considered method of finding the distance from a given point to some plane is especially convenient when, in addition to the distance d, it is also required to determine the coordinates of the point M_0 (x_0, y_0, z_0), that is, the coordinates of the foot of the perpendicular dropped from the point M on the plane p. To find the distance d only, it is more convenient to make use of a ready formula which we are going to derive. Let M_1 (x_1, y_1, z_1) be an arbitrary point of the plane p. The distance d is equal to the absolute value of the projection of the vector $\overrightarrow{M_1M} = \{x_M - x_1, y_M - y_1, z_M - z_1\}$ on the direction of the normal vector of the plane $\mathbf{N} = \{A, B, C\}$. Since $\mathbf{N} \cdot \overrightarrow{M_1M} = |\mathbf{N}| \operatorname{pr}_n \overrightarrow{M_1M}$, we have

$$d = |\operatorname{pr}_n \overrightarrow{M_1M}| = \frac{|\mathbf{N} \cdot \overrightarrow{M_1M}|}{|\mathbf{N}|} = \frac{|A(x_M - x_1) + B(y_M - y_1) + C(z_M - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$
$$= \frac{|Ax_M + By_M + Cz_M - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}},$$

where n is an axis parallel to the vector $\mathbf{N}$.

The point M_1 (x_1, y_1, z_1) belongs to the plane p, therefore the equality $Ax_1 + By_1 + Cz_1 + D = 0$ is valid, whence $Ax_1 + By_1 + Cz_1 = -D$. Now, we finally find:

$$d = \frac{|Ax_M + By_M + Cz_M + D|}{\sqrt{A^2 + B^2 + C^2}}. \tag{4}$$

Using formula (4) for solving Example 1, we can find at once that

$$d = \frac{|4 \cdot 0 + \sqrt{11} \cdot 0 + 3 \cdot 0 + 150|}{\sqrt{16 + 11 + 9}} = \frac{150}{\sqrt{36}} = 25.$$

Example 2. The area of the lateral surface and the volume of a regular rectangular pyramid are respectively equal to S and V. Find the distance from a vertex of the base of the pyramid to the plane of the lateral face not containing this vertex.

△ We arrange the Cartesian coordinate system in the way shown in Fig. 42 and denote by a the length of the edge at the base of the pyramid and by H its altitude. Find the distance d from the point O (0, 0, 0) to the plane p containing the face CDB. It is easy to see that the plane p passes through the points C (0, a, 0), B ($-a$, a, 0), and $D\left(-\frac{a}{2}, \frac{a}{2}, H\right)$, that is, the equation of this plane has the form (see subsection 1 of 3.1):

$$\begin{vmatrix} (x - x_C) & (y - y_C) & (z - z_C) \\ (x_B - x_C) & (y_B - y_C) & (z_B - z_C) \\ (x_D - x_C) & (y_D - y_C) & (z_D - z_C) \end{vmatrix} = \begin{vmatrix} x & (y - a) & z \\ -a & 0 & 0 \\ -\frac{a}{2} & -\frac{a}{2} & H \end{vmatrix}$$
$$= aHy + \frac{a^2}{2} z - a^2 H = 0.$$

The distance d from the point O (0, 0, 0) to the found plane is computed by formula (4):

$$d=\frac{\left|0\cdot 0+aH\cdot 0+\frac{a^2}{2}\cdot 0-a^2H\right|}{\sqrt{0^2+a^2H^2+\frac{a^4}{4}}}=\frac{aH}{\sqrt{H^2+\frac{a^2}{4}}}.$$

The volume V of the pyramid is equal to one-third of the product of the area of the base a^2 by the altitude H, that is, $V=\frac{1}{3}a^2H$. The

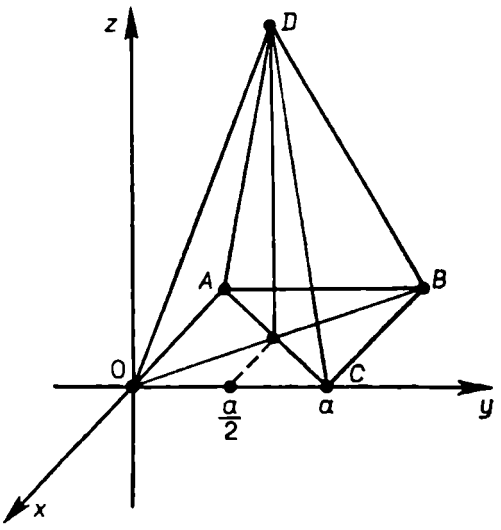

Fig. 42

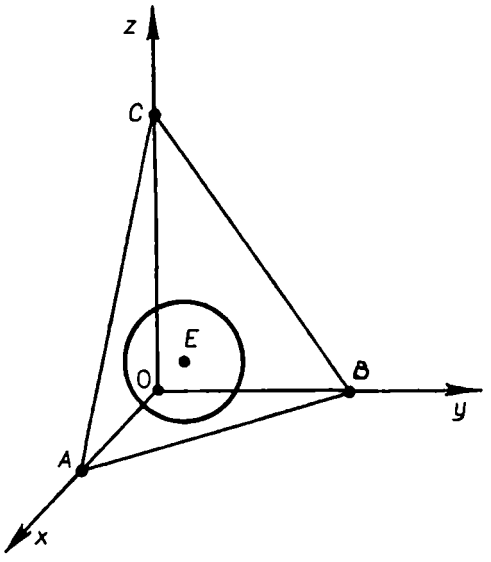

Fig. 43

area of the lateral surface S is equal to four areas of one lateral face, that is,

$$S=4\cdot\frac{a}{2}\sqrt{H^2+\frac{a^2}{4}}=2a\sqrt{H^2+\frac{a^2}{4}},$$

then

$$d=\frac{aH}{\sqrt{H^2+\frac{a^2}{4}}}=\frac{6V}{S}.$$

We have succeeded in expressing the desired distance d in terms of the known quantities V and S. ▲

Example 3. The base of the pyramid $COAB$ is a right triangle OAB. The edge CO is perpendicular to the plane of the base. The lengths of the edges OA, OB, and OC are equal to 2, 3, and 4, respectively. Find the radius of the sphere inscribed in the pyramid.

△ We arrange the Cartesian coordinate system as shown in Fig. 43. Then the centre of the inscribed sphere will lie at the point $E(R, R, R)$, where R is the radius of the inscribed sphere. The distance d from the point E to the plane of the face ABC must be equal to R. Let us find the equation of the plane ABC passing through the points A (2, 0, 0), B (0, 3, 0), and C (0, 0, 4):

$$\begin{vmatrix} x-2 & y & z \\ -2 & 3 & 0 \\ -2 & 0 & 4 \end{vmatrix} = 12(x-2)+8y+6z = 12x+8y+6z-24 = 0.$$

The distance from the point $E(R, R, R)$ to this plane is found by formula (4):

$$d = \frac{|12R+8R+6R-24|}{\sqrt{12^2+8^2+6^2}} = \frac{|13R-12|}{\sqrt{61}}.$$

Equating this distance to the radius R, we find that $\sqrt{61}R = |13R-12|$, whence either $R = \frac{12}{13+\sqrt{61}}$ or $R = \frac{12}{13-\sqrt{61}} > 2$, but the second root does not suit since in this case the point $E(R, R, R)$ lies outside the pyramid. Thus, $R = \frac{12}{13+\sqrt{61}}$. ▲

Exercises

1. Show that the volume of the parallelepiped constructed on the diagonals of the faces of a given parallelepiped having a common vertex is twice the volume of the given parallelepiped.

2. The vertices of a pyramid are located at the points A (2, 1, −1), B (3, 0, 1), C (2, −1, 3), and D (0, −7, 0). Find the altitude of the pyramid dropped from the vertex D.

3. Find the distance from the origin to the plane:

(a) $4x - 2y - 4z + 7 = 0$;

(b) $15x + 16y - 12z - 100 = 0$;

(c) $\sqrt{2}x + y - z + 32 = 0$.

4. Find the distance from the given point to the given plane:

(a) M (1, 2, 4), $2x + 2y - z - 11 = 0$;

(b) M (7, 0, −7), $18x - 6y + 9z + 14 = 0$;

(c) $M(\sqrt{5}, \sqrt{12}, 2)$, $y\sqrt{3} + z + 20 = 0$.

5. Compute the distance from the point M (0, 1, −3) to the plane passing through the points M_1 (3, 1, −5), M_2 (8, 3, 3), M_3 (−2, −1, 4).

6. Compute the distance between two parallel lines:

(a) $3x - 6y - 2z + 35 = 0$ and $6x - 12y - 4z + 5 = 0$,

(b) $5x + 2y - 3z - 5 = 0$ and $10x + 4y - 6z + 5 = 0$,

(c) $7x - y + \sqrt{2}z - 3\sqrt{2} = 0$ and $7\sqrt{2}x - \sqrt{2}y + 2z - 6 = 0$.

7. The edge of a cube is a. Find the length of the line segment joining the midpoints of two skew edges.

3.4. Second-order Surfaces

1. The General Equation of the Second Order in Three Variables. It was shown in the preceding section that if there is a Cartesian coordinate system, then any equation of the first degree in three variables

$$Ax + By + Cz + D = 0, \tag{1}$$

where the coefficients A, B, and C are not all zero, is the equation of a plane.

Now, we shall consider equations of the second degree. The general equation of the second degree in three variables has the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Kz + L = 0, \tag{2}$$

where $A^2 + B^2 + C^2 + D^2 + E^2 + F^2 \neq 0$.

Surfaces described by second-degree equations are called second-order surfaces.

In the theory of second-order surfaces which is beyond the scope of the present course of higher mathematics it is proved that it is always possible to change to a new coordinate system in which equation (2) will have one of the following seventeen forms:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$;

(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$;

(4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$; (5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$; (6) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

(7) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; (8) $y^2 = 2px$; (9) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$;

(10) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$; (11) $y^2 - a^2 = 0$; (12) $y^2 = 0$;

(13) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$; (14) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$;

(15) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$; (16) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$; (17) $y^2+a^2=0$,

where the coefficients a, b, c, and p are not equal to zero. Equations 1-17 are called canonical.

Equations 15-17 correspond to empty sets since they have no real solutions. Equation 14 describes the only point in space O (0, 0, 0). Equation 13 describes the z-axis of a coordinate system since for any point of this axis $x = y = 0$. Equation 12 describes one coordinate plane, namely, the xz-plane. Equation 11 defines two planes: $y - a = 0$ and $y + a = 0$ which are parallel to the xz-plane of the coordinate system; equation 10 also describes two planes $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$.

Equations 1-9 correspond to new geometrical objects: ellipsoid, hyperboloid of two sheets, hyperboloid of one sheet, elliptic paraboloid, hyperbolic paraboloid, elliptic cylinder, hyperbolic cylinder, parabolic cylinder, and cone (enumerated in the increasing order of their equation numbers).

2. The Elliptic, Hyperbolic, and Parabolic Cylinders. The canonical equations of the elliptic cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

hyperbolic cylinder

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \tag{2}$$

and parabolic cylinder

$$y^2 = 2px \tag{3}$$

coincide in form, respectively, with the equations of the ellipse, hyperbola, and parabola lying in the xy-plane. These equations do not contain the variable (coordinate) z. Therefore if a point M_1 $(x_1, y_1, 0)$ satisfies equations (1)-(3), then these equations are also satisfied by points with coordinates (x_1, y_1, z), where z is any number. Thus, equations (1)-(3) describe cylindrical surfaces with rectilinear directrices parallel to the z-axis. Figure 44 shows the shape of these cylinders.

3. The Ellipsoid. To investigate the shape of the ellipsoid, let us consider its canonical equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{1}$$

It follows from equation (1) that the coordinate planes serve as symmetry planes of the ellipsoid, and the origin as the centre of symmetry.

The numbers a, b, c are called the semiaxes of the ellipsoid and represent the lengths of the line segments from the origin to the points of intersection of the ellipsoid with the coordinate axes. The ellipsoid represents a bounded surface enclosed, as is seen from equation (1), in a parallelepiped $|x| \leqslant a$, $|y| \leqslant b$, $|z| \leqslant c$. We can

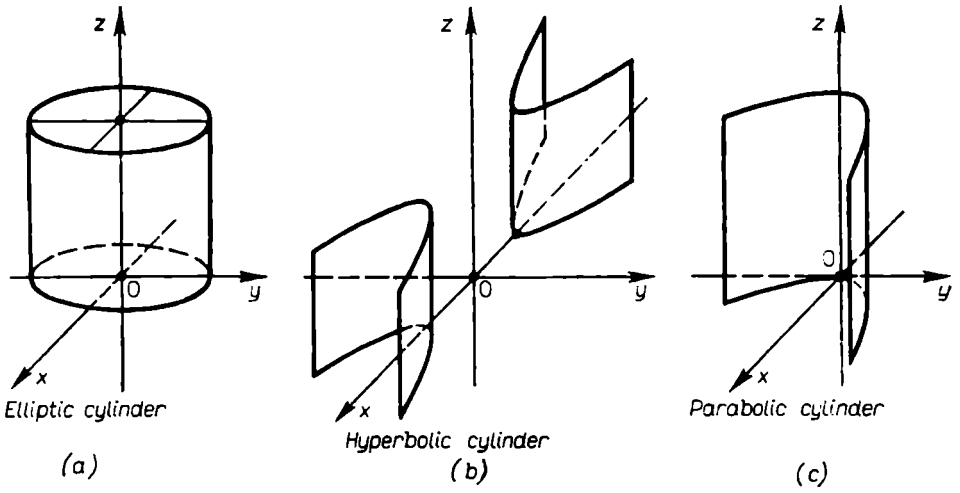

Fig. 44

get an additional idea as to the shape of the ellipsoid by investigating it using the method of sections. Let us first determine the line of intersection of the ellipsoid with the plane

$$z = h, \quad \text{where } -c \leqslant h \leqslant c, \tag{2}$$

which is parallel to the xy-plane. The line of intersection of surfaces (1) and (2) is given by the system of these two equations:

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1, \\ z = h. \end{cases}$$

This system of equations is equivalent to the system

$$\begin{cases} \dfrac{x^2}{(a\sqrt{1-h^2/c^2})^2} + \dfrac{y^2}{(b\sqrt{1-h^2/c^2})^2} = 1, \\ z = h. \end{cases} \tag{3}$$

Hence it is seen that the section of the ellipsoid by the plane $z = h$ coincides with the section of the elliptic cylinder by the same plane and is an ellipse with semiaxes $a\sqrt{1 - h^2/c^2}$ and $b\sqrt{1 - h^2/c^2}$. As $|h|$ increases, that is, as the cutting plane $z = h$ moves off from the central coordinate plane xOy (its equation is $z = 0$), the dimensions of the ellipses in the sections decrease. The same is true for the

sections by the planes $y = h$ and $x = h$, only the lengths of the semiaxes of the ellipses obtained in section being different. A schematic representation of the ellipse is given in Fig. 45.

In the particular cases $a = b \neq c$, $a = c \neq b$, and $b = c \neq a$, the ellipsoid is called the ellipsoid of revolution since in these cases

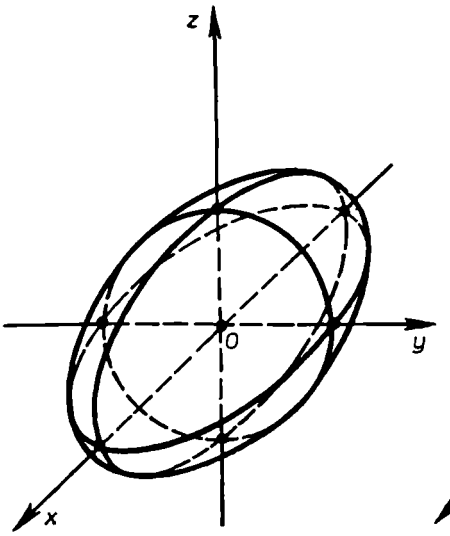

Fig. 45

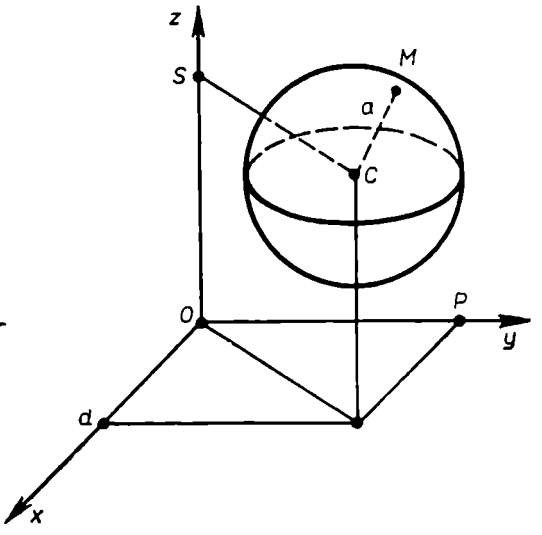

Fig. 46

it can be generated by rotating the ellipse about its axes of symmetry. For example, the ellipsoid of revolution

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$$

is obtained by rotating the ellipse $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ (lying in the xz-plane) about the z-axis. If $a = b = c$, then equation (1) takes the form:

$$x^2 + y^2 + z^2 = a^2. \tag{4}$$

Equation (4) defines the set of points in space equidistant from the origin. Thus, to equation (4) there corresponds the sphere (spherical surface) of radius a with centre at the origin.

The equation

$$(x - d)^2 + (y - p)^2 + (z - s)^2 = a^2 \tag{5}$$

defines the set of points in space equidistant from the point C (d, p, s), that is, it defines the sphere of radius a with centre at the point C (d, p, s) (Fig. 46). Equation (4) is the canonical equation of the sphere, and equation (5) the general equation of the sphere.

Example 1. Write the equation of the sphere of radius $R = 3$ with centre at the point C $(2, -5, 1)$.

△ Substituting the values of the coordinates of the point C and the value of the radius in formula (5), we get $(x-2)^2+(y-(-5))^2+(z-1)^2=3^2$ or $(x-2)^2+(y+5)^2+(z-1)^2=9$. ▲

Example 2. Prove that the equation $x^2+y^2+z^2-2x+4y-6z+5=0$ is the equation of the sphere. Find its centre and radius.

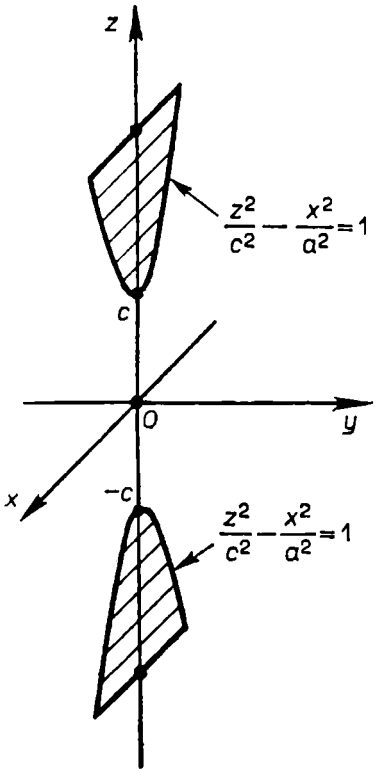

Fig. 47

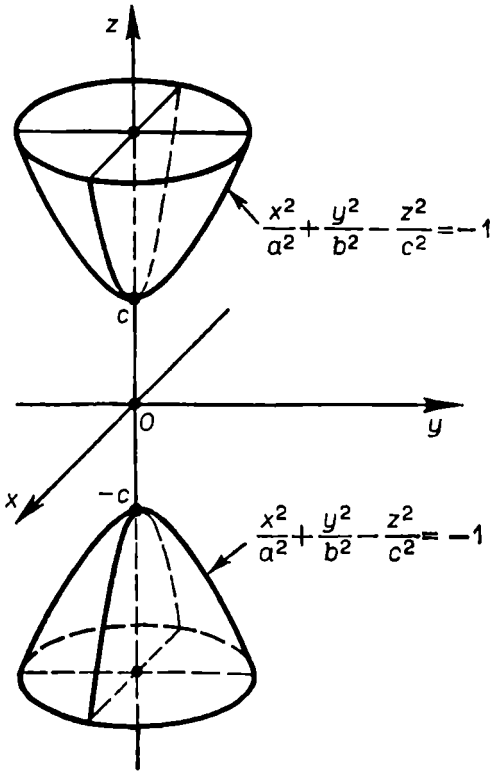

Fig. 48

△ We transform the left-hand side of the given equation: $x^2-2x+1-1+y^2+4y+4-4+z^2-6z+9-9+5=0$ or $(x-1)^2+(y+2)^2+(z-3)^2=3^2$. This equation describes the sphere of radius $R=3$ centred at the point $C(1,-2,3)$. ▲

4. The Hyperboloid of Two Sheets. To investigate the shape of the hyperboloid of two sheets, let us consider its canonical equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1. \tag{1}$$

It follows from equation (1) that the coordinate planes are the planes of symmetry of the hyperboloid of two sheets, and the origin is its centre of symmetry.

In the particular case, when $a = b \neq c$ equation (1) defines the hyperboloid of revolution of two nappes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = -1, \tag{2}$$

which can be generated by rotating about the z-axis the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1$ $\left(\text{or, which is the same, } \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1\right)$, lying in the xz-plane (Fig. 47). The hyperboloid of revolution of two nappes (2) is shown in Fig. 48. The planes $z = h$, $|h| \geqslant c$ cut this surface in circles since the lines of intersection are defined by the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = -1, \\ z = h \end{cases}$$

or by the equivalent system

$$\begin{cases} x^2 + y^2 = a^2\left(\frac{b^2}{c^2} - 1\right), \\ z = h. \end{cases}$$

Since the hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ generating this surface consists of two isolated branches (see Fig. 47), the surface itself also consists of two isolated nappes.

In the general case, when $a \neq b \neq c$, the hyperboloid of two sheets (1) has practically the same shape as it is shown in Fig. 48. In this case, the planes $z = h$, $|h| \geqslant c$ cut the surface not in circles but in ellipses defined by the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \\ z = h \end{cases}$$

or the equivalent system

$$\begin{cases} \frac{x^2}{(a\sqrt{h^2/c^2 - 1})^2} + \frac{y^2}{(b\sqrt{h^2/c^2 - 1})^2} = 1, \\ z = h. \end{cases}$$

The planes $x = h$, $|h| \geqslant 0$ or $y = h$, $|h| \geqslant 0$ cut hyperboloid of two sheets (1) in hyperbolas defined, for instance, by the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \\ y = h \end{cases}$$

or the equivalent system

$$\begin{cases} \dfrac{z^2}{(c\sqrt{h^2/b^2+1})^2} - \dfrac{x^2}{(a\sqrt{1+h^2/b^2})^2} = 1, \\ y = h. \end{cases}$$

5. The Hyperboloid of One Sheet. The hyperboloid of revolution of two nappes was generated by rotating the hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ (Fig. 47) about the axis of symmetry Oz intersecting the hyperbola. If now we rotate the same hyperbola about the other axis of symmetry (the axis Ox) which does not intersect the hyperbola, we shall obtain the surface

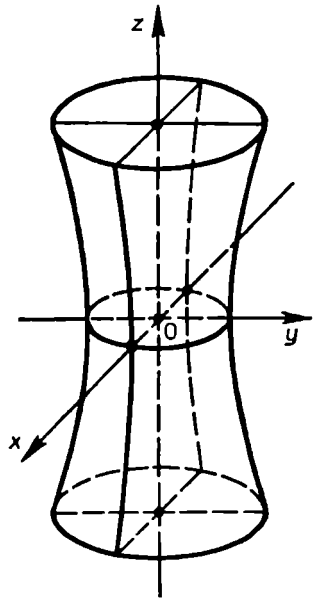

Fig. 49

$$-\frac{x^2}{a^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = 1, \qquad (1)$$

which is called the hyperboloid of revolution of one nappe oriented along the x-axis. But it is more convenient to represent the hyperboloid of revolution of one nappe

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1,$$

oriented along the z-axis (Fig. 49). It is generated by rotating the hyperbola $\frac{y^2}{a^2} - \frac{z^2}{c^2} = 1$ lying in the yz-plane about the axis of symmetry Oz which does not intersect the hyperbola. When rotated, each branch of the hyperbola generates the same surface which, thus, consists of one nappe. This circumstance just determines the name of the surface. In the case of the general equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (2)$$

the hyperboloid of one sheet looks like one represented in Fig. 49, the only difference being that now the planes $z = h$ cut the hyperboloid not in circles but in ellipses defined by the system of equations

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, \\ z = h \end{cases}$$

or by the equivalent system

$$\begin{cases} \dfrac{x^2}{(a\sqrt{1+h^2/c^2})^2} + \dfrac{y^2}{(a\sqrt{1+h^2/c^2})^2} = 1, \\ z = h. \end{cases}$$

Hyperboloid of one sheet (2) is cut by the planes $x = h$ or $y = h$ in hyperbolas defined, for instance, by the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \\ y = h \end{cases}$$

or by the equivalent system

$$\begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} = \left(1 - \frac{h^2}{b^2}\right), \\ y = h. \end{cases}$$

We can see from the last system that the hyperbolas cut by the planes $y = h$, $|h| < b$ and those cut by the planes $y = h$, $|h| > b$ are oriented in space in different way. In the first case they are intersected by the axis of symmetry parallel to the x-axis, while in the second case by the axis of symmetry parallel to the z-axis. In the intermediate case, that is, in the section by the planes $y = \pm b$, we get a pair of straight lines $\frac{x}{a} - \frac{z}{c} = 0$ and $\frac{x}{a} + \frac{z}{c} = 0$.

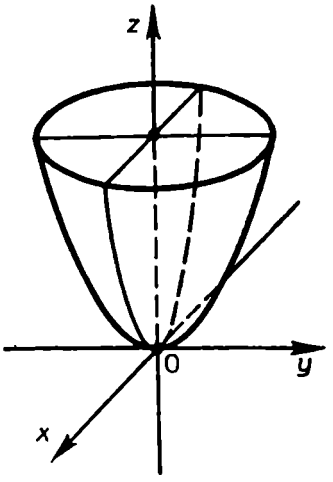

Fig. 50

6. The Elliptic Paraboloid. Consider the parabola $z = \frac{x^2}{a^2}$, lying in the xz-plane. When rotated about the z-axis, this parabola generates the surface

$$z = \frac{x^2}{a^2} + \frac{y^2}{a^2}, \qquad (1)$$

called the paraboloid of revolution (Fig. 50). The elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0 \qquad (2)$$

looks almost in the same way as the paraboloid of revolution represented in Fig. 50, the only difference being that the planes $z = h$, $h \geqslant 0$ cut it not in circles but in ellipses defined by the system of equations

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0, \\ z = h \end{cases}$$

or by the equivalent system

$$\begin{cases} \frac{x^2}{(a\sqrt{h})^2} + \frac{y^2}{(b\sqrt{h})^2} = 1, \\ z = h. \end{cases}$$

The coordinate planes xOz and yOz are the planes of symmetry of elliptic paraboloid (2).

7. The Hyperbolic Paraboloid. The canonical equation of the hyperbolic paraboloid has the form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0. \tag{1}$$

Let us investigate the shape of the hyperbolic paraboloid using the

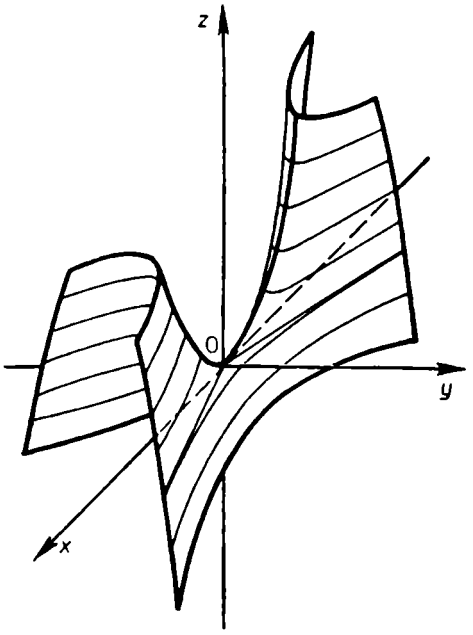

Fig. 51

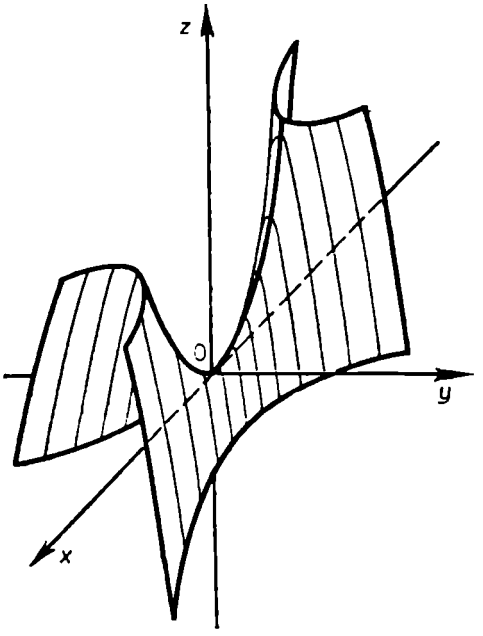

Fig. 52

method of sections. The lines of intersection of the hyperbolic paraboloid with the planes $z = h$ represent for $h > 0$ hyperbolas

$$\begin{cases} \dfrac{x^2}{(a\sqrt{h})^2} - \dfrac{y^2}{(b\sqrt{h})^2} = 1, \\ z = h, \end{cases}$$

and for $h < 0$ hyperbolas oriented in a different way:

$$\begin{cases} \dfrac{y^2}{(b\sqrt{|h|})^2} - \dfrac{x^2}{(a\sqrt{|h|})^2} = 1, \\ z = h. \end{cases}$$

The plane $z = 0$ cuts the hyperbolic paraboloid in straight lines

$$y = \pm \frac{b}{a} x.$$

The lines of intersection of hyperbolic paraboloid with the planes $x = h$ and $y = h$ for any h are parabolas

$$\begin{cases} z = \dfrac{h^2}{a^2} - \dfrac{y^2}{b^2}, \\ x = h \end{cases} \quad \text{and} \quad \begin{cases} z = -\dfrac{h^2}{b^2} + \dfrac{x^2}{a^2}, \\ y = h. \end{cases}$$

Schematic representation of the hyperbolic paraboloid is given in Figs. 51 and 52. The first of these figures also shows the hyperbolic sections by the planes $z = h$, and the second the parabolic sections by the planes $x = h$.

8. The Cone. The canonical equation of the cone has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \tag{1}$$

The point O (0, 0, 0) belonging to the cone is called the vertex of the cone. Let us make sure that this cone is generated by straight lines passing through its vertex. Let a point M_1 (x_1, y_1, z_1), other than the origin, belong to cone (1), that is, its coordinates satisfy the equality

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 0. \tag{2}$$

Then any point M_2 of the straight line l, passing through the points O and M_1, also belongs to the cone. Indeed, the radius vectors $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$ are collinear (they lie on the same straight line). Then the equality $\overrightarrow{OM_2} = \lambda\overrightarrow{OM_1}$ is valid, where $\lambda \neq 0$ (see 1.1). Hence we find that $\overrightarrow{OM_2} = \{\lambda x_1, \lambda y_1, \lambda z_1\}$, that is, the point M_2 has the coordinates $(\lambda x_1, \lambda y_1, \lambda z_1)$, which, by virtue of equality (2), satisfy the equation of cone (1).

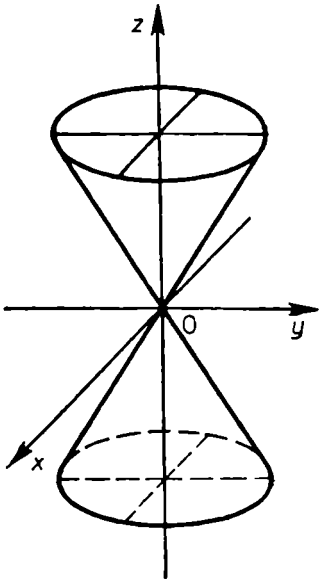

Fig. 53

Thus, the cone is generated by straight lines (elements) passing through the origin. Figure 53 represents schematically the cone. We can easily see that cone (1) is cut by the planes $z = h$ in ellipses

$$\begin{cases} \dfrac{x^2}{(ah/c)^2} + \dfrac{y^2}{(bh/c)^2} = 1, \\ z = h. \end{cases}$$

In the particular case $a = b$, we have an ordinary circular cone known from school geometry.

Second-order surfaces are widely used in engineering. For example, the first television tower constructed in Moscow according to the design of V.G. Shukhov has the shape of the hyperboloid of revolution of one nappe. The metal reflectors of lighting units have the shape of the paraboloid of revolution, and so forth.

Exercises

1. Find the centre and radius of the given sphere:

(a) $x^2 + (y - 1)^2 + (z + 3)^2 = 81$;
(b) $(x - 2)^2 + (y + 4)^2 + (z - 1)^2 = 72$.

2. Prove that the given equation is the equation of the sphere:

(a) $x^2 + y^2 + 2y + z^2 + z - 1 = 0$;
(b) $x^2 + y^2 + z^2 - 8x + 4y + 2z - 43 = 0$.

3. What is the line of intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane:

(a) $y = 1$; (b) $y = \frac{1}{2}$; (c) $y = 2$; (d) $x = 0$; (e) $z = -\frac{1}{2}$?

4. Write the equation of the surface of revolution generated by the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ rotating about the y-axis.

5. Write the equation of the surface of revolution generated by the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ rotating about the z-axis. Sketch the surface.

6. Write the equation of the surface of revolution generated by the parabola $y^2 = 6z$ about the z-axis. Sketch the surface.

7. What surfaces are given by the equations:

(a) $x^2 + y^2 = 25$; (b) $x^2 + y^2 + z^2 = 25$;

(c) $\frac{x^2}{9} + \frac{y^2}{4} = 1$; (d) $\frac{x^2}{9} - \frac{y^2}{4} = 1$; (e) $y^2 = 7x$?

(8) Write the equation of the conical surface generated by rotating of two intersecting straight lines $\frac{y^2}{9} - \frac{z^2}{16} = 0$ and $x = 0$ about the z-axis.

9. Determine what surface is described by the equation $14x = y^2 + z^2$ and sketch it.

10. Determine what surface is given by the equation $\frac{x^2}{48} - \frac{y^2}{15} - \frac{z^2}{10} = 1$ and find the straight line along which it intersects with the plane $z = 1$.

11. Determine what surface is given by the equation $\frac{x^2}{9}-\frac{y^2}{16}+\frac{z^2}{4}=1$ and find the straight line along which it intersects with the plane $z+2=0$.

12. Determine what surface is given by the equation $x^2+y^2-4z=0$ and find the line along which it intersects with the plane $x-2=0$.

Part 2

MATHEMATICAL ANALYSIS

Chapter 4

REAL AND COMPLEX NUMBERS

4.1. Fundamentals of Set Theory

1. A Set and Its Elements (Members). Subsets. The set is defined as a collection of objects united by some common feature. For instance, the set of school children, the set of the letters of an alphabet, the set of natural numbers, the set of points in a straight line, etc. The objects constituting the set are called its members or elements. Sets are usually denoted by capital letters of the Latin alphabet (A, B, C, . . .), while elements of sets by lower-case letters of the Latin or Greek alphabet (a, b, c, α, β, γ, . . .).

The notation $\alpha \in A$ means that the member α belongs to the set A. The notation $\alpha \notin A$ means that the member α does not belong to the set A. If N is the set of natural numbers, then $1 \in N$, $0 \notin N$.

A set is considered to be specified (known) if either all of its members are enumerated or such a property of its members is pointed out that enables us to judge whether or not a given member belongs to the set. Such properties are known as characteristic properties. For instance, speaking of the set of all even numbers, we indicate the characteristic property of its members: each number belonging to this set is exactly divisible by two.

The sets consisting of the same members are said to be equal. If the sets A and B are equal to each other, then we write $A = B$. If any member in a set B is also a member of a set A, then B is called a subset of A. In this case, we say that the set B is contained in the set A or A contains B and write: $B \subset A$ or $A \supset B$.

We also consider the set which contains no members. Such a set is called the empty set and is denoted by the symbol $\varnothing$. By definition, the empty set is a subset of any set. Consequently, any set A always has two obvious subsets: A and $\varnothing$.

Let the set A consist of the numbers 1, 3, and 5 which is written in the form $A = \{1, 3, 5\}$. Then the sets $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$, $\varnothing$, $\{1, 3, 5\}$ are subsets of the set A.

The sets A and B are said to be put into one-to-one correspondence if to each member of the set A there corresponds the only member of the set B, and, conversely, to each member of the set B there corre-

sponds the only member of the set A. The sets A and B that can be put into one-to-one correspondence are called equivalent; we then write $A \sim B$. For example, the set of natural numbers N and the set of all even numbers M are equivalent sets, that is, $N \sim M$. Indeed, it is easy to establish one-to-one correspondence between the sets N and M if every natural number $n \in N$ is associated with an even number $m = 2n \in M$ and vice versa, if every even number $m = 2n \in M$ is associated with $n \in N$. It is obvious that if $A \sim B$ and $B \sim C$, then $A \sim C$.

A set A is called finite if there exists a natural number n such that the set A is equivalent to the set $\{1, 2, \ldots, n\}$. In this case the set A is said to contain n members.

There are two methods of ascertaining equivalence of two finite sets: (1) to ascertain directly one-to-one correspondence between their members; (2) to count the members of the given sets and to compare their numbers. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{\text{square, triangle, rectangle, trapezoid}\}$, then these are equivalent finite sets. This follows from the fact that both sets contain four members each.

A finite set consisting of n members is called ordered if its members are numbered in some manner by natural numbers $1, 2, \ldots, n$. For instance, the set of pupils of a form can be ordered in the following way: their names can be numbered in the register by natural numbers (the first on the list, the second on the list, and so forth).

2. Intersection and Union of Sets. Consider the set of natural numbers multiple of the number 2, and the set of natural numbers

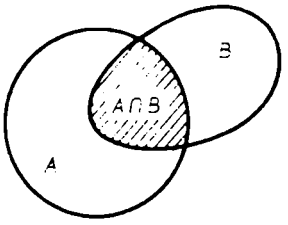

Fig. 54

Fig. 55

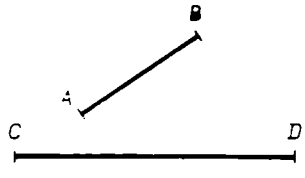

Fig. 56

multiple of the number 5. Then the set of numbers multiple of the number 10 consists of the members which enter into each of the two sets under consideration.

The set C consisting of all those and only those members which belong to either of the given sets A and B is called the intersection of the sets A and B and is denoted as $A \cap B$ ($\cap$ is the sign of intersection). Fig. 54 illustrates an intersection of the sets A and B. For point sets (for instance, geometrical figures) the meaning of the term "intersection of sets" corresponds to that of the customary term "intersection of figures". Thus, the intersection of the line segments AB and CD (Fig. 55) is a line segment AD.

Two sets whose intersection is an empty set are called disjoint sets. For instance, the intersection of the line segments AB and CD (Fig. 56) is empty, therefore they are disjoint sets.

The union of two sets A and B is defined as such a set C which consists of all the members of the sets A and B, and only of them. In this case we write: $A \cup B$ ($\cup$ is the sign of union). If the sets A and B have common members (that is, $A \cap B \neq \varnothing$), then each of the common members is taken in the set C only once. For example

(a) $\{1, 2, 5\} \cup \{3, 4\} = \{1, 2, 3, 4, 5\}$,

(b) $\{1, 2, 4\} \cup \{3, 4\} = \{1, 2, 3, 4\}$.

3. Subtraction of Sets. Complement of a Set. Direct Product of Two Sets. Let there be given two sets A and B. The set C which

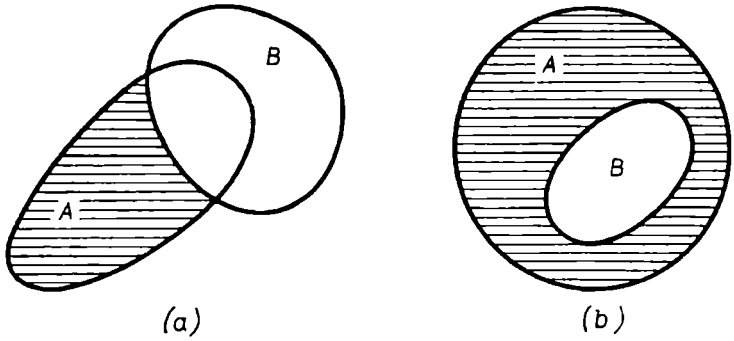

Fig. 57

consists of all members of the set A not belonging to the set B is called the difference of the sets A and B and is denoted as $C = A \setminus B$ (in Fig. 57, *a* the difference of the sets is indicated by hatching). For example:

(a) if $A = \{1, 3, 5, 6\}$, $B = \{1, 3\}$, then $A \setminus B = \{5, 6\}$;

(b) if $A = \{1, 4, 6\}$, $B = \{3, 5\}$, then $A \setminus B = \{1, 4, 6\}$;

(c) if $A = \{2, 3\}$, $B = \{1, 2, 3\}$, then $A \setminus B = \varnothing$.

If $A \supset B$, then the difference $A \setminus B$ is called the complement of the set B to the set A (Fig. 57, *b*). Note that the result of the operation "complement" depends essentially on the set to which a given set is "complemented". For example, the set of all fractional numbers is the complement of the set of integers to the set of all rational numbers; but if we consider the complement of the set of integers to the set of real numbers, then the complement of this set will be the set of all fractional and all irrational numbers.

The direct product of two sets A and B is the set whose members are all ordered pairs (x, y) in which the first component is a member of A and the second component a member of B. The direct product of the sets A and B is denoted by $A \times B$.

Example. Let $A = \{2, 1, 3, 9\}$, $B = \{2, 4, 5\}$. Find $A \times B$.

△ We have $A \times B = \{(2, 2); (2, 4); (2, 5); (1, 2); (1, 4); (1, 5); (3, 2); (3, 4); (3, 5); (9, 2); (9, 4); (9, 5)\}$. ▲

To describe the direct product of two sets, it is often convenient to use the "geometrical language" in which the members of the set $A \times B$ are called points. For example, if $Z(x, y)$ is a point of the number plane, then $x \in A$ is called the abscissa, and $y \in B$ the ordinate of the point Z. This terminology is clear if we pay attention to the fact that the set of points in the plane is essentially a direct product of the form $\boldsymbol{R} \times \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of real numbers.

Exercises

1. Let $A = \{-3, -2, -1, 0, 1, 2, 3, 7\}$, $B = \{5, 3, 2, 1, 0, -2, -3\}$, $C = \{-4, -3, \ldots, 3, 4\}$. Find the sets $A \cup B$, $A \cap B$, $A \cup C$, $A \cap C$, $B \cup C$.

2. Let $\boldsymbol{N}$ be the set of natural numbers, $\boldsymbol{Z}$ the set of integers, and the sets A, B and C are defined in Exercise 1. Find $A \cap \boldsymbol{N}$, $B \cap \boldsymbol{Z}$, $A \cup \boldsymbol{N}$, $B \cup \boldsymbol{Z}$, $\boldsymbol{N} \cap \boldsymbol{Z}$, $(A \cap B) \cap \boldsymbol{N}$.

3. Let M be the set of values of the expression $4 + 7b$ for $b = -2, 0, 3$. Write all the subsets of M.

4. Read the following notations: $a \in A$, $a \notin A$, $C \subset D$, $\varnothing \subset A$, $F \subset E$, $A \sim B$, $A = B$, $A \cup B = C$, $A \cap B = D$, $A \setminus B = E$, $A \times B = L$.

5. Let $\boldsymbol{N}$ be the set of natural numbers, $\boldsymbol{Z}$ the set of integers, $\boldsymbol{Q}$ the set of rational numbers and $\boldsymbol{R}$ the set of real numbers. How are these sets interrelated?

6. Let F_1 be the set of all possible parallelograms, F_2 the set of rectangles, F_3 the set of rhombuses, and F_4 the set of squares. Write down the result of the following operations: (a) $F_2 \cap F_3$, (b) $F_2 \cup F_3 \cup F_4 \cup F_1$.

7. Indicate the empty sets among the following: (a) the set of integral roots of the equation $x^2 - 16 = 0$; (b) the set of integral roots of the equation $x^2 + 16 = 0$; (c) the set of natural numbers less than 1.

8. Let $A = \{5, 7, 8\}$, $B = \{8, 9\}$. Find $A \cap B$, $A \cup B$, $A \setminus B$, $A \times B$.

4.2. The Set of Real Numbers

1. Rational Numbers. The idea of numbers was formed by man gradually under the influence of practical needs. For instance, the concept of natural numbers was developed in connection with the necessity to count objects, that is, with the necessity to answer the question: "How many members (elements) does a given finite set contain?" On counting the books on the shelves of a bookcase, we say that there are six books on the first shelf, that is, the set of the books

on the first shelf contains six members (books), and there are seven books on the second shelf, that is, the set of the books on the second shelf contains seven members (books), and so forth. However, the following situation may also occur: one of the shelves of the bookcase is free of books (it is occupied, say, with copy-books or other objects), that is, the set of books on this shelf is an empty set since none of the members (books) belongs to it. In this case the number zero makes it possible to answer the question: "How many members are contained in a given set?"

Adding the number 0 to the set of all natural numbers $N = \{1, 2, 3, \ldots\}$, we get the set of nonnegative integral numbers $Z_0 = \{0, 1, 2, 3, \ldots\} = N \cup \{0\}$.

But nonnegative integers alone turned out to be insufficient for solving the problems set by practice, and, hence, the mathematical problems reflecting a given real situation. In order to characterize air temperature above and below zero, as also motion in opposite directions, we need opposite numbers (or the so-called opposites of numbers). For instance, air temperature of six degrees above zero and six degrees below zero is written as $+6$ °C and -6 °C, respectively. The numbers 5 and -5 are called opposites of each other: -5 is the opposite of 5, and 5 is the opposite of -5. In the general case, $-n$ will be the opposite of the natural number n, and n will be the opposite of the number $-n$. The number zero is assumed to be the opposite to itself.

Natural numbers, zero, and opposites of natural numbers constitute the set Z of integers, that is $Z = N \cup \{0\} \cup N_-$, where N_- is the set of numbers opposite to natural.

Measurement of quantities led to the necessity of extending the set of integers, of introducing fractional numbers. For instance, the height of the Spasskaya tower of the Moscow Kremlin is 71 m, the diameter of the clock-face of the Kremlin chime of bells amounts to 6.12 m, etc. Here is another example: a teaching period lasts 45 minutes or $\frac{3}{4}$ hour, and one of the intervals 20 minutes or $\frac{1}{3}$ hour.

Integers and fractional numbers form the set Q of rational numbers. Any rational number can be written in the form of a fraction $\frac{m}{n}$, where $m \in Z$ and $n \in N$ (that is, the numerator is an integer, while the denominator is a natural number). A rational number can be expressed by various fractions. For example,

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{10}{20}, \quad -\frac{2}{3} = \frac{-2}{3} = \frac{-20}{30} = -\frac{24}{36};$$

$$-1.2 = \frac{-6}{5} = \frac{-12}{10} = -\frac{120}{100}; \quad 5 = \frac{5}{1} = \frac{10}{2} = \frac{50}{10};$$

$$0 = \frac{0}{1} = \frac{0}{6} = \frac{0}{10}.$$

Among the fractions representing a given rational number, there is always only one irreducible fraction, for integers this is a fraction whose denominator is equal to 1.

Let there be given a rational number $\frac{m}{n}$. Dividing the numerator by the denominator, we get a finite (terminating) or infinite (nonterminating) decimal fraction. For example,

$$\frac{1}{2}=0.5; \quad \frac{1}{3}=0.3333\ldots;$$

$$-\frac{7}{5}=-1.4; \quad \frac{-3}{7}=-0.428571428571\ldots.$$

Let us agree to represent a finite decimal fraction in the form of an infinite decimal fraction in which, to the right of nonzero decimal digits, zeros are added in place of subsequent decimal digits. For example,

$$\frac{1}{2}=0.5=0.5000\ldots, \quad -\frac{7}{5}=-1.4000\ldots.$$

Let us also agree to write whole numbers in the form of a nonterminating decimal fraction with zeros standing to the right of the decimal point in places usually occupied by decimal digits. For example,

$$45=45.000\ldots, \quad -3=-3.000\ldots.$$

Thus, any rational number is representable in the form of a nonterminating decimal fraction:

$$a_0 \cdot a_1 a_2 a_3 \ldots,$$

where a_0 is an integer, and each of a_1, a_2, a_3, ... is one of the digits 0, 1, 2, ..., 9.

Definition. An infinite decimal $a_0 \cdot a_1 a_2 a_3 \ldots$ is called repeating (periodic) if there exist natural numbers N and p such that $a_{n+p} = a_n$ for any $n \geqslant N$.

For example, 0.3333 ... is a repeating decimal since, setting $N = 1$ and $p = 1$, we have $a_{n+1} = a_n = 3$ for any $n \geqslant 1$. The number 0.5000 ... is also an infinite repeating decimal for which $a_{n+1} = a_n = 0$ for $n \geqslant 2$, that is, $N = 2$ and $p = 1$. The number $-3.476353535\ldots$ is also represented in the form of an infinite repeating decimal since, putting $N = 4$ and $p = 2$, we get $a_{n+2} = a_n$ for any $n \geqslant 4$. Indeed, for any $n \geqslant 4$ we have $a_n = 3$ for even n and $a_n = 5$ for odd n.

Infinite repeating decimals are customarily written in the following way: the decimal 8.4900 ... is written as 8.49(0), the decimal 0.333 ... as 0.(3), the decimal $-3.476353535\ldots$ can be written as $-3.476(35)$. The number enclosed in the parentheses is called the

period. The decimal 8.49(0) is read as "eight point fourty-nine and zero repeating"; the decimal 0.(3) as "zero point and three repeating", and the decimal —3.476(35) as "minus three point four, seven, six, and thirty-five repeating".

Any rational number can be represented in the form of an infinite repeating decimal. For example, the rational number $\frac{7}{11}$ can be represented in the form of the repeating decimal 0.(63). To get such a representation, it is sufficient to divide 7 by 11:

$$\begin{array}{r|l} & 0.63 \\ \hline 11 & 7.000 \\ & 66 \\ \hline & 40 \\ & 33 \\ \hline & 7 \end{array}.$$

As soon as we obtain the remainder equal to 7, the further calculation may be ceased since the remainders and the digits in the quotient will repeat. Therefore, $\frac{7}{11} = 0.6363 \ldots = 0.63$, that is, we have zero point and sixty-three in the period. In the general case, with an arbitrary rational number $\pm\frac{m}{n}$, where m and n are natural numbers, we proceed in a similar manner, dividing m by n. Since in dividing by n the remainder can attain only n possible values 0, 1, 2, . . ., $n - 1$, no more than after n steps the decimal digits in the quotient will begin repeating which means that the division of m by n yields a repeating decimal. We can show that any infinite repeating decimal that has the digit 9 in the period is equal to an infinite repeating decimal with the period equal to zero in which the decimal digit which precedes the period is enlarged by unity as compared with the corresponding digit of the initial decimal fraction.

Example. Show that $0.4(9) = 0.5(0)$.

△ Let $x = 0.4(9)$. Then $100x - 10x = 49.(9) - 4.(9) = 45$, whence $x = \frac{45}{90} = \frac{1}{2} = 0.5 = 0.5(0)$. ▲

Let us agree that in further representation of rational numbers in terms of finite repeating decimals we shall exclude from consideration infinite repeating decimals whose period is equal to 9.

2. Real Numbers. The set of all infinite (nonterminating) decimals is called the set of real numbers and is denoted by $\boldsymbol{R}$. Note that the set $\boldsymbol{Q}$ of all rational numbers is a subset of the set $\boldsymbol{R}$. Real numbers which are not rational are called irrational.

Theorem. *There is no rational number whose square is equal to the number* 2.

□ We shall prove this theorem using the method of proof by contradiction. Let us assume that there exists a rational number whose square is equal to 2 and which is representable by an irreducible fraction $\frac{m}{n}$. Then we have:

$$\left(\frac{m}{n}\right)^2 = 2 \text{ or } m^2 = 2n^2. \tag{1}$$

Thus, m^2 is an even number. But then the number m is also even. Indeed, suppose that $m = 2k + 1$ (that is, odd), then $m^2 = (4k^2 + 4k) + 1$ would be odd since $(4k^2 + 4k)$ is even. But if m is an even number, then $m = 2k$, and formula (1) implies that

$$n^2 = 2k^2,$$

that is, the number n^2 is even, and, hence, the number n is also even. Consequently, our supposition has led us to the following result: both numbers m and n turn out to be even. But this contradicts the original assumption of the irreducibility of the fraction $\frac{m}{n}$. This contradiction shows that our assumption on the existence of a rational number whose square is equal to 2 was incorrect. ■

The number $\sqrt{2}$ is irrational: $\sqrt{2} = 1.41421356 \ldots$.

Another example of an irrational number is the number π which is used to denote the ratio of the circumference of a circle to its diameter: $\pi = 3.1415926 \ldots$.

3. The Absolute Value (or Modulus) of a Real Number. The absolute value (modulus) of a real number x is denoted by $|x|$ and is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases} \tag{1}$$

For example, if $x = 2$, then $|x| = 2$, if $x = -2.37$, then $|x| = 2.37$.

Consider a horizontal straight line in the plane. We choose on it the positive direction (in Fig. 58—from left to right) which will be

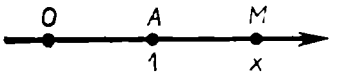

Fig. 58

denoted by an arrow, we then choose the origin (point O), and the unit line segment OA. The straight line OA is called the number line.

Any real number x can be represented by a point M on the straight line OA by constructing a line segment OM of length $|x|$ laying it off to the right from the point O if $x > 0$ and to the left from the

point O if $x < 0$. The number zero is represented by the point O. Therefore to each real number there corresponds quite a definite point on the number line. The converse is also true: to each point of the number line there corresponds a certain real number (represented just by this point). The set $\boldsymbol{R}$ of real numbers is often called the number line, and the real numbers themselves the points of this line.

Consider the basic properties of the absolute value of a real number. It follows from the definition that $|x| \geqslant 0$ for any x, the equality to zero occurring only if $x = 0$. The following obvious inequality is valid:

$$-|x| \leqslant x \leqslant |x|. \tag{2}$$

Theorem 4.2.1. *The absolute value of a product (quotient) is equal to the product (quotient) of the absolute values of the factors (dividend and divisor), that is,*

$$|x \cdot y| = |x| \cdot |y|; \tag{3}$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \quad y \neq 0. \tag{4}$$

□ Let us prove formula (3). Let $x > 0$ and $y > 0$. According to the definition of the absolute value, we get $|x| = x$ and $|y| = y$. Since $xy > 0$, by the definition, we have $|xy| = xy$. Consequently,

$$|xy| = |x| \cdot |y|.$$

Let now $x < 0$ and $y < 0$. Then $(-x) > 0$, $(-y) > 0$ and $|x| = -x$, $|y| = -y$. Using the foregoing, we have

$$|xy| = |(-x) \cdot (-y)| = (-x) \cdot (-y) = |x| \cdot |y|.$$

It remains to consider the case when the signs of x and y are opposite. Let, for definiteness, $x > 0$ and $y < 0$. Since $xy < 0$ and $|x| = x$, $|y| = -y$, using definition (1), we have

$$|xy| = -(xy) = x \cdot (-y) = |x| \cdot |y|.$$

Formula (3) has been proved. Formula (4) is proved in a similar way. ■

Setting in formula (3) $y = -1$, we find

$$|x| = |-x|, \tag{5}$$

that is, the absolute values of two opposite numbers are equal to each other. On the number line, opposite numbers are represented by points lying at the same distance from the point O but on different sides from it (Fig. 59).

Theorem 4.2.2. *The inequality*

$$|x| \leqslant \alpha, \tag{6}$$

where $\alpha > 0$, *is equivalent to the double inequality*

$$-\alpha \leqslant x \leqslant \alpha. \tag{7}$$

□ Let inequality (6) be satisfied. If $x \geqslant 0$, then $|x| = x$, and the right-hand inequality in double inequality (7) is satisfied by virtue of (6), while the left-hand inequality is obvious since $-\alpha < 0$ and $x \geqslant 0$.

If $x < 0$, then $|x| = -x$, and inequality (6) takes the form $-x \leqslant \alpha$. Multiplying the last inequality by -1, we see that the

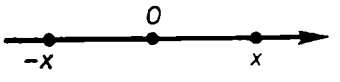

Fig. 59

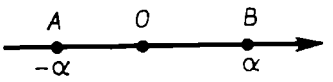

Fig. 60

left-hand inequality in formula (7) is satisfied, while the right-hand inequality is obvious since $x < 0$ and $\alpha > 0$. Thus, we have proved that inequality (6) implies double inequality (7).

Let us now assume that double inequality (7) is satisfied. If $x \geqslant 0$, then $x = |x|$, and the validity of inequality (6) follows from the right-hand inequality of (7). If $x < 0$, then $x = -|x|$, and inequality (7) takes the form $-\alpha \leqslant -|x| \leqslant \alpha$. Multiplying the left-hand inequality in the last double inequality by -1, we get: $|x| \leqslant \alpha$. ■

Note that on the number line the numbers satisfying inequality (6) are represented by the points lying between the points A and B (Fig. 60).

Theorem 4.2.3. *The inequality*

$$|x| \geqslant \alpha \tag{8}$$

implies that

$$x \geqslant \alpha \quad or \quad x \leqslant -\alpha. \tag{9}$$

□ If $x \geqslant 0$, then $|x| = x$, and inequality (8) implies that $x \geqslant \alpha$. If $x < 0$, then we have $|x| = -x$, and relation (8) takes the form $-x \geqslant \alpha$. Multiplying the last inequality by -1, we find that $x \leqslant -\alpha$. ■

Example 1. Solve the inequality $|x - 3| \leqslant 5$.

△ According to formula (7), this inequality is equivalent to the double inequality $-5 \leqslant x - 3 \leqslant 5$, therefore $-2 \leqslant x \leqslant 8$. ▲

Example 2. Solve the inequality $(x + 4)^2 \leqslant 9$.

△ Taking the square root, we get the inequality $|x + 4| \leqslant 3$ which is equivalent to the double inequality $-3 \leqslant x + 4 \leqslant 3$, whence $-7 \leqslant x \leqslant -1$. ▲

Further, let us dwell on other theorems characterizing the properties of the absolute value.

Theorem 4.2.4. *The absolute value of a sum is less than, or equal to, the sum of the absolute values of the terms, that is,*

$$|x+y| \leqslant |x| + |y|. \tag{10}$$

□ According to formula (2), for any x and y we have:

$$-|x| \leqslant x \leqslant |x|,$$
$$-|y| \leqslant y \leqslant |y|.$$

Adding together the inequalities, we get:

$$-(|x| + |y|) \leqslant (x+y) \leqslant (|x| + |y|).$$

According to Theorem 4.2.2 from the last inequality we get:

$$|x+y| \leqslant |x| + |y|. \blacksquare$$

Theorem 4.2.5. *The absolute value of a difference is greater than, or equal to, the difference of the absolute values of the minuend and subtrahend, that is,*

$$|x-y| \geqslant |x| - |y|. \tag{11}$$

□ By virtue of the inequality for the absolute value of a sum

$$|x| = |(x-y)+y| \leqslant |x-y| + |y|,$$

whence there follows formula (11). ■

Exercises

1. What members of the set $A = \{-40, -32.4, -8, -\frac{1}{9}, 0, \frac{5}{7}, 6, 12, 19\frac{2}{9}, 30\}$ are natural numbers, integers (whole numbers), fractional numbers, rational numbers, negative numbers, non-negative numbers?

2. Form the subsets of the set $B = \{-24, -23\frac{1}{3}, -22, -9, 0, \frac{1}{5}, 2, 5, 9, 10, 12, 24\}$, whose members are: (a) natural numbers; (b) integers (whole numbers); (c) fractions; (d) odd numbers; (e) numbers multiple of 4; (f) negative numbers.

3. Which of the following statements are true: $N \subset Z_0$, $Z_0 \subset N$, $N \subset Z$, $Z \subset N$, $Z \subset Q$, $Q \subset Z$.

4. Represent the following numbers as a ratio of an integer to a natural number: (a) -30, -14, 0, 6, 25, 85; (b) $-3\frac{1}{8}$, -1.6, $-2\frac{4}{5}$, 0.6, $4\frac{1}{9}$, $9\frac{11}{15}$.

5. Write the following numbers in the form of infinite decimals: (a) -8; (b) 9; (c) $-\frac{1}{6}$; (d) -13.85; (e) -0.5; (f) 6.1; (g) $\frac{7}{9}$; (h) $\frac{1}{6}$; and (i) $\frac{11}{30}$.

6. Which of the numbers $-\frac{5}{9}$, $\frac{27}{12}$, $-\frac{6}{7}$, 9, -2.3, 3, 5, 0, 0.(2), 0.(4), 1.(3) are rational? Represent each rational number as the ratio $\frac{m}{n}$, where $m \in \boldsymbol{Z}$, $n \in \boldsymbol{N}$.

4.3. Sets of Numbers. Intervals. A Neighbourhood of a Point

Consider concrete subsets of the set of real numbers.

1. Intervals. The set of all real numbers x satisfying the double inequality $a \leqslant x \leqslant b$ is called a closed interval with the points a and b as endpoints, and is denoted as $[a, b]$. The length of the closed interval $[a, b]$ is the number equal to $b - a$.

The set of all real numbers x such that $a < x < b$ is called an open interval with a and b as endpoints and is denoted by (a, b). The length of the open interval is the number equal to $b - a$.

The set of all real numbers x such that $a \leqslant x < b$ is termed a half-open interval and is denoted by $[a, b)$. The interval $(a, b]$ is defined in a similar manner, that is, the set of numbers x satisfying the inequality $a < x \leqslant b$.

The set of all $x > a$ is said to be an infinite interval and is designated as $(a, +\infty)$. The infinite intervals $[a, +\infty)$, $(-\infty, b)$, and $(-\infty, b]$ are defined by the inequalities $x \geqslant a$, $x < b$, and $x \leqslant b$, respectively.

The set $\boldsymbol{R}$ is sometimes called an infinite interval, the corresponding notation being $(-\infty, +\infty)$. The interval $(0, +\infty)$ is denoted by $\boldsymbol{R}_+$, and the interval $(-\infty, 0)$ by $\boldsymbol{R}_-$.

Definition. The neighbourhood of a number c is defined as any interval (a, b) containing c, while the ε-neighbourhood (read: "epsilon neighbourhood") of the point c as the interval $(c - \varepsilon, c + \varepsilon)$, where $\varepsilon > 0$.

2. Bounded and Unbounded Sets of Numbers.

Definition 1. A set of numbers A is said to be bounded above (below) if there is a real number M (number m) such that for each member x of the set the inequality $x \leqslant M$ ($x \geqslant m$) is satisfied. In this case, the number M (number m) is called the upper (lower) bound of the set A.

A set of numbers A bounded above has infinitely many upper bounds. Indeed, if M_1 is an upper bound of the set A, then any real number $M_2 > M_1$ is also an upper bound of the set A since for every

member x of the set A the inequality $x \leqslant M_1 \leqslant M_2$ is satisfied, that is, $x \leqslant M_2$. A similar note can be made with respect to lower bounds of a set bounded below.

Definition 2. The least of all upper bounds of a set of numbers A bounded above is called the upper bound of this set. The greatest of all lower bounds of a set of numbers B bounded below is termed the lower bound of this set. Sometimes, the upper (lower) bound of a set is called the least upper (the greatest lower) bound of the set.

For example, the set of negative real numbers has the upper bound—the number zero, but this number does not belong to this set. The set of natural numbers N has the lower bound $x = 1$ which belongs to this set. Thus, the upper (lower) bound of a set can either belong or not belong to this set.

Definition 3. A set of numbers A is said to be bounded if it is bounded both below and above. Note that any closed interval is a bounded set since the left-hand endpoint of this interval serves as the lower bound, its right-hand endpoint serving as the upper bound.

Infinite sets of numbers have not a single bound. Any unbounded interval may serve as an example of an infinite set.

3. The Number Plane. It was indicated above that real numbers can be represented by points of the number line. Analogously, ordered pairs of real numbers can be represented by points of the coordinate plane. Therefore it is expedient that the set of ordered pairs of real numbers is called the number plane, any ordered pair being referred to as a point of the number plane. We shall denote the number plane by the symbol R^2. Just as in the case of the number line, we may apply the geometrical terminology to the number plane. For instance, the set of pairs $(x, y) \in R^2$ or points whose coordinates satisfy the equation $y = x$ is a straight line, viz. the bisector of the first and third quadrants. The set of points $(x, y) \in R^2$ whose coordinates satisfy the equation $y = x^3$ is a cubical parabola.

Example. Using the geometrical language, describe the set of points (x, y) satisfying the equation $(x - 2)(y + 4) = 0$.

△ Since either $x - 2 = 0$ or $y + 4 = 0$, the given set is a union of the sets of points of two straight lines: $x = 2$ and $y = -4$. ▲

Exercises

1. On the number line, indicate the set of points x satisfying the given relation:

(a) $|x| = 2$; (b) $|x| \leqslant 3$; (c) $|x| \geqslant 8$; (d) $|x| > 9$;

(e) $|x| = \sqrt{3}$; (f) $|x| > \sqrt{2}$; (g) $|x| \leqslant \sqrt{5}$, (h) $|x| \geqslant \frac{\sqrt{7}}{3}$.

2. Using the geometrical language, describe and represent on the coordinate plane the following sets:

(a) $(x-3)(y-3)=0$; (b) $(x+4)(y+5)=0$;

(c) $\frac{x-y}{y-1}=0$; (d) $x>0$, $y\leqslant 0$, (e) $x>0$, $y\leqslant 5$;

(f) $x<0$, $y\geqslant -2$.

3. Point out which of the intervals listed below are ε-neighbourhoods of the point x:

(a) $(x-5, x+2)$; (b) $(x-0.1, x+0.1)$;

(c) $(x+1, x+2)$; (d) $(-\infty, x+1)$; (e) $(x-1, +\infty)$;

(f) $[x-2, x+2]$.

4. Find the bound points of the following intervals:

(a) $[2, 5]$; (b) $\left[-7, \frac{3}{2}\right]$; (c) $(-4, -1)$; (d) $(-\infty, 2]$;

e) $\left[-11\frac{1}{3}, +\infty\right)$; (f) $(-\infty, -3)$; (g) $(-\infty, +\infty)$.

4.4. The Set of Complex Numbers

1. Complex Numbers. The set of real numbers is insufficient for solving any algebraic equation. Indeed, quadratic equations with negative discriminants (for instance: $x^2+1=0$, $x^2+x+1=0$, etc.) have no solution on the set of real numbers. To overcome this difficulty, we introduce the set of complex numbers which include the set of real numbers as a subset. Since we want the equation $x^2+1=0$ to have a solution on the set of complex numbers, it is necessary to introduce a certain new number and regard it as the solution of this equation. We shall denote this new number by the symbol (letter) i and call it the imaginary unit. Thus, for the number i the equation $i^2+1=0$ (or $i^2=-1$) holds true. Further, the set of real numbers should be supplemented with new numbers bi which are called imaginary or pure imaginary, and are regarded as products of real numbers b by the imaginary unit i. Finally, we shall write the sum of a real number a and an imaginary number bi in the form $a+bi$ (or, which is the same $a+ib$).

Definition. Complex numbers are defined as expressions of the form $a+bi$ (a and b are real numbers, i the symbol of imaginary unit), for which the concept of equality and operations of addition and multiplication are introduced in the following way:

(a) two complex numbers a_1+b_1i and a_2+b_2i are equal if and only if $a_1=a_2$ and $b_1=b_2$;

(b) the sum of the numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is defined as the number $a_1 + a_2 + (b_1 + b_2) i$;

(c) the product of the numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is defined as the number $a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$.

Thus, complex numbers are added and multiplied by the formulas

$$(a_1 + b_1 i) + (a_2 + b_2 i) = a_1 + a_2 + (b_1 + b_2) i, \tag{1}$$

$$(a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i. \tag{2}$$

Setting in formula (2) $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, we get the relation

$$ii = i^2 = -1, \tag{3}$$

which ensures the existence of the solution of the equation $x^2 + 1 = 0$. The student needs not memorize formula (2) since it is obtained automatically by formally multiplying the binomials $a_1 + b_1 i$ and $a_2 + b_2 i$ according to the ordinary rule for multiplying binomials and replacing -1 for i^2 in accordance with formula (3).

Complex numbers are often denoted by one letter, the letters z or w being usually used for this purpose; sometimes they are used with subscripts, for instance, z_1, z_2, w_0. The notation $z = a + bi$ just means that the complex number $a + bi$ is denoted by the letter z.

It is easy to make certain that the newly introduced operations of addition (1) and multiplication (2) possess the following properties:

1. Commutativity of addition:

$$z_1 + z_2 = z_2 + z_1.$$

2. Associativity of addition:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

3. For any complex numbers z_1 and z_2 there is a complex number z such that $z_1 + z = z_2$. This number is called the difference of the numbers z_2 and z_1 and is denoted as $z = z_2 - z_1$.

4. Commutativity of multiplication:

$$z_1 z_2 = z_2 z_1.$$

5. Associativity of multiplication:

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

6. For any complex numbers $z_1 \neq 0 + 0i$ and z_2 there is a number z such that $z_1 z = z_2$. This number is called the quotient of the complex numbers z_2 and z_1 and is denoted by $z = \frac{z_2}{z_1}$. Division by the complex number $0 + 0i$ which is called zero is impossible.

7. Distributivity:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

All the above listed properties of the operations of addition and multiplication follow directly from formulas (1) and (2) defining these operations and the equality of complex numbers. As an example, let us prove property 7.

□ Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, and $z_3 = a_3 + b_3 i$. We compute the left-hand side of equality (4):

$$z_1 (z_2 + z_3) = (a_1 + b_1 i)(a_2 + a_3 + (b_2 + b_3) i)$$
$$= a_1 (a_2 + a_3) - b_1 (b_2 + b_3) + (b_1 (a_2 + a_3) + a_1 (b_2 + b_3)) i$$
$$= a_1 a_2 + a_1 a_3 - b_1 b_2 - b_1 b_3 + (b_1 a_2 + b_1 a_3 + a_1 b_2 + a_1 b_3) i. \quad (5)$$

We compute the right-hand side of equality (4):

$$z_1 z_2 + z_1 z_3 = (a_1 + b_1 i)(a_2 + b_2 i) + (a_1 + b_1 i)(a_3 + b_3 i)$$
$$= a_1 a_2 - b_1 b_2 + (a_1 b_2 + b_1 a_2) i + a_1 a_3 - b_1 b_3 + (a_1 b_3 + b_1 a_3) i$$
$$= a_1 a_2 - b_1 b_2 + a_1 a_3 - b_1 b_3 + (b_1 a_2 + b_1 a_3 + a_1 b_2 + a_1 b_3) i. \quad (6)$$

Relations (5) and (6) imply the equality $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$. Thus, we have proved property 7. ■

The validity of properties 1-7 means that addition and multiplication of complex numbers obey the same laws as addition and multiplication of real numbers.

Moreover, the newly introduced operations of addition and multiplication enable us to consider complex numbers as a generalization of real numbers and to regard real numbers as a particular case of complex numbers. Indeed, let us consider not all complex numbers, but only those of the form $a + 0i$. From formulas (1) and (2) it is obvious that the results of addition and multiplication (the same is true for subtraction and division) are numbers of the same form. Besides, we see that the rules of operations on complex numbers of the form $a + 0i$ completely coincide with the corresponding rules for operations on real numbers. These two circumstances just make it possible to identify the complex number $a + 0i$ with the real number a and to hold that $a + 0i = a$. For example, $0 + 0i = 0$, $1 + 0i = 1$, $-1 + 0i = -1$.

The set of all complex numbers is denoted by C. The set of real numbers R is a subset of the set of complex numbers, that is, $R \subset C$. The real number a is called the real part of a complex number $a + bi$. The real number b is termed the imaginary part of the complex number $a + bi$.

By definition, two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$, that is, if their real and imaginary parts coincide. Thus, one equality $z_1 = z_2$ of complex numbers is equivalent to two equalities $a_1 = a_2$, $b_1 = b_2$ of real numbers. One more remark: the notions "greater than" and

"less than" cannot be defined for complex numbers. Therefore the notations $z > 0$, $4 + 2i < 2 + 4i$, and the like are meaningless.

The numbers $a + bi$ and $a - bi$, that is, two numbers which differ from each other only by the sign of the imaginary part are called the conjugate complex numbers. The conjugate of the number z is denoted by $\bar{z}$. Applying formulas (1) and (2), we find

$$z + \bar{z} = (a + bi) + (a - bi) = 2a, \tag{7}$$

$$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2, \tag{8}$$

that is, the sum $z + \bar{z}$ and the product $z \cdot \bar{z}$ of conjugate complex numbers are real numbers.

Example 1. Given two complex numbers $z_1 = -1 + 3i$ and $z_2 = 5 + 2i$. Find their sum and difference.

$$\triangle\ z_1 + z_2 = (-1 + 3i) + (5 + 2i) = -1 + 3i + 5 + 2i$$
$$= 4 + 5i.$$
$$z_2 - z_1 = (5 + 2i) - (-1 + 3i) = 5 + 2i + 1 - 3i$$
$$= 6 - i. \blacktriangle$$

Example 2. Given two complex numbers $z_1 = 1 + i$ and $z_2 = 2 - i$. Find their product and quotient.

$\triangle\ z_1 z_2 = (1 + i)(2 - i) = 2 - i + 2i - i^2 = 2 + i + 1 = 3 + i$. Multiplying the numerator and denominator of the fraction $\frac{z_2}{z_1} = \frac{2-i}{1+i}$ by the same complex number $(1 - i)$, conjugate to the denominator of this fraction, we get:

$$\frac{z_2}{z_1} = \frac{(2-i)(1-i)}{(1+i)(1-i)} = \frac{2-i-2i-1}{1+1} = \frac{1-3i}{2} = \frac{1}{2} - \frac{3}{2}i. \blacktriangle$$

Example 3. Find the complex number $z = \frac{(2-i)(1+2i)}{1+i}$.

$\triangle$ Multiplying the numbers in the numerator, we get:

$$z = \frac{2+4i-i-2i^2}{1+i} = \frac{4+3i}{1+i}.$$

We multiply the numerator and denominator of the fraction $\frac{4+3i}{1+i}$ by the number conjugate to the denominator, that is, by $(1 - i)$. We then get:

$$z = \frac{(4+3i)(1-i)}{(1+i)(1-i)} = \frac{4-4i+3i-3i^2}{1+1} = \frac{7-i}{2} = \frac{7}{2} - \frac{1}{2}i. \blacktriangle$$

2. The Modulus of a Complex Number.

Definition. The modulus (or the absolute value) of the complex number $z = a + bi$ is denoted by $|z|$ and is defined by the formula

$$|z| = \sqrt{a^2 + b^2}. \tag{1}$$

For example, if $z = 1 - 2i$, then $|z| = \sqrt{1+4} = \sqrt{5}$, if $z = -2 + 3i$, then $|z| = \sqrt{4+9} = \sqrt{13}$. It follows from the definition that the modulus of a complex number is a real number, the following inequality being satisfied for it:

$$|z| \geqslant 0. \tag{2}$$

Note that the equality to zero occurs only if $z = 0$, that is, $a = b = 0$. Comparing formula (1) with formula (8) from subsection 1, we find

$$|z| = |\bar{z}| = \sqrt{z\bar{z}}, \tag{3}$$

that is, the moduli of conjugate complex numbers are equal to each other.

It is easy to prove that

$$|z_1 z_2| = |z_1| \cdot |z_2|. \tag{4}$$

□ By formula (2) from the preceding subsection we have $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$. By the definition of the modulus,

$$\begin{aligned} |z_1 z_2|^2 &= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \\ &= a_1^2 a_2^2 + 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2) \\ &= (a_1^2 + b_1^2)(a_2^2 + b_2^2) = |z_1|^2 \cdot |z_2|^2, \end{aligned}$$

whence follows formula (4). ■

Similarly, we prove that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 = 0. \tag{5}$$

Example. Given two complex numbers $z_1 = -1 + 3i$ and $z_2 = 5 + 2i$. Compute the modulus of their product.

△ We compute $|z_1| = \sqrt{1+9} = \sqrt{10}$. Similarly, $|z_2| = \sqrt{25+4} = \sqrt{29}$. By formula (4), we have $|z_1 z_2| = |z_1| \cdot |z_2| = \sqrt{10}\sqrt{29} = \sqrt{290}$. ▲

For a real number $z = a + 0i$ the modulus coincides with the absolute value of the number: $|z| = |a + 0i| = \sqrt{a^2} = |a|$.

Exercises

1. Find the sum and product of the complex numbers z_1 and z_2:
(a) $z_1 = 5 + 4i$, $z_2 = -2 + 3i$; (b) $z_1 = -8 - 7i$, $z_2 = -3i$;
(c) $z_1 = 5 + \sqrt{3}i$, $z_2 = 5 - \sqrt{3}i$.

2. Find the difference $z_2 - z_1$ and the quotient $\frac{z_2}{z_1}$ if:

(a) $z_1 = 1 + i$, $z_2 = 1 - i$; (b) $z_1 = 1 + 2i$, $z_2 = 5$;

(c) $z_1 = a - \sqrt{b}i$, $z_2 = a + \sqrt{b}i$; (d) $z_1 = 3$, $z_2 = i$.

3. Compute:

(a) $\frac{1+i}{1-i} + \frac{1-i}{1+i}$; (b) $\left(\frac{-1+\sqrt{3}i}{2i}\right)^2$; (c) $i^5 + i^6 + i^7 + i^8$.

4. Find the real and imaginary part of the given complex numbers:

(a) $z = \frac{(1+2i)^3}{i} + i^7$; (b) $z = (2-i)^2(3+11i)$; (c) $z = \frac{2+3i}{1-5i} + i^4$.

5. Find the complex numbers:

(a) $z = i + \frac{4i+3}{1-7i}$; (b) $z = (2+i)^6$;

(c) $z = \frac{5+13i}{8-2i} + \frac{(1+2i)^2}{1+i}$.

6. Prove the given equalities: (a) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$; (b) $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$.

7. Solve the equations: (a) $(i - z)(1 + 2i) + (1 - iz)(3 - 4i) = 1 + 7i$; (b) $z^2 + \overline{z} = 0$.

8. Compute the modulus of the given number:

(a) $z = (-1+i)(2-i)$; (b) $z = 147(2+3i)$;

(c) $z = \frac{1+i}{1-i} + \frac{1-i}{1+i}$.

4.5. The Complex Plane. The Argument of a Complex Number. Subsets of Complex Numbers

1. The Complex Plane. When studying complex numbers, we can successfully use geometrical terminology and concepts if every complex number $a + bi$ is represented by a point $M(a, b)$ of the coordinate plane such that its abscissa is equal to the real part of the complex number, and its ordinate to the imaginary part. On the other hand, to every point $M(a, b)$ of the coordinate plane there corresponds the complex number $a + ib$ (Fig. 61). In this case, the coordinate plane is called the complex plane. The axis of abscissas is then called the real axis since it incorporates the points corresponding to complex numbers $a + 0i$, that is, to real numbers. The axis of ordinates is called the imaginary axis: lying on it are the points corresponding to pure imaginary complex numbers $0 + bi$.

As is seen from Fig. 61, the quantity introduced earlier—the modulus of the complex number $|z| = |a + bi| = \sqrt{a^2 + b^2}$—

is equal to the length of the vector $\overrightarrow{OM}$, that is, the vector emanating from the origin O (0, 0) and terminating at the point M (a, b).

2. The Argument of a Complex Number. Complex numbers z, having one and the same modulus $|z| = r$, obviously, correspond to the points of the complex plane lying on the circle of radius r with centre at the point $z = 0$, that is, at the point O (Fig. 62). If

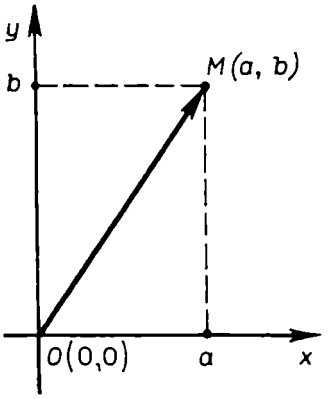

Fig. 61

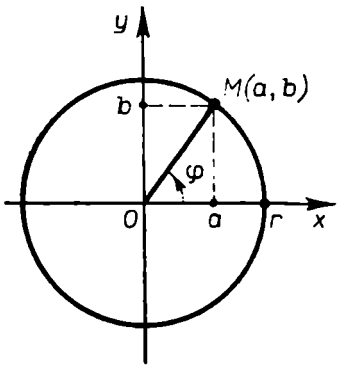

Fig. 62

$|z| \neq 0$, then infinitely many numbers exist with the given modulus. Only one complex number, namely $z = 0$, has a modulus equal to zero.

Geometrically, it is obvious that in order to single a concrete number out of the set of complex numbers with a given modulus $|z| = r \neq 0$, it suffices to specify the direction of the vector $\overrightarrow{OM}$ (to specify, say, the angle φ, see Fig. 62).

Definition. The angle between the positive direction of the real axis and the radius vector $\overrightarrow{OM}$ with the origin at the point O (0, 0) and terminus at the point M (a, b) is called the argument (or amplitude) of a complex number $z = a + bi \neq 0$. The angle is regarded as positive if it is measured anticlockwise, and as negative if it is measured clockwise.

The argument of a complex number $z = a + bi$ is denoted by $\arg z$ or $\arg (a + bi)$. By specifying the modulus and argument, a complex number is defined completely. For the number $z = 0$ the argument is not defined; in this and only this case the number is given only by its modulus.

From Fig. 62 we find that given the modulus r and argument φ of the complex number $z = a + bi$, its real part a and imaginary part b are defined by the formulas

$$a = r \cos \varphi, \quad b = r \sin \varphi. \tag{1}$$

The argument of a complex number, unlike its modulus, is defined not uniquely. Indeed, if the angle φ is the argument of a number z, then each of the angles $\varphi_k = \varphi + 2\pi k$, $k \in Z$ will also be an argument of this number. Any two arguments of a complex number differ from each other by a term multiple of 2π. According to formula (1), this ambiguity does not affect the magnitudes of the real and imaginary parts of the complex number.

Example 1. Given the modulus of a complex number $r = 2$ and its argument $\varphi = \frac{\pi}{6}$. Find the real and imaginary part of the complex number.

△ By formula (1), we have $a = r\cos\varphi = 2\cos\frac{\pi}{6} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$, $b = r\sin\varphi = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1$. ▲

Example 2. Given a complex number $z = 1 + i$. Find its modulus and argument.

△ We first compute the modulus of the number z: $|z| = r = \sqrt{1+1} = \sqrt{2}$. Now, we find the argument of the number z

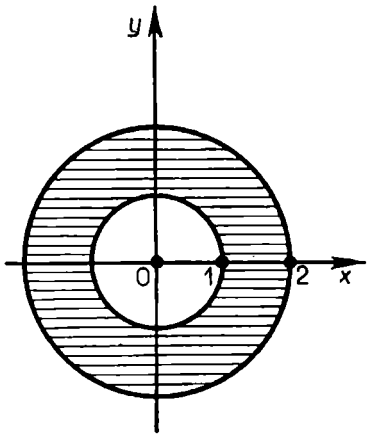

Fig. 63

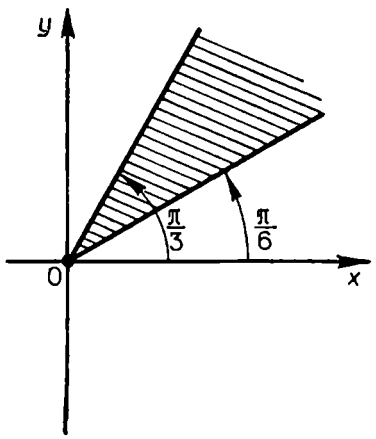

Fig. 64

whose real part is equal to unity. Making use of formula (1) and the found value of the modulus, we get: $1 = r\cos\varphi = \sqrt{2}\cos\varphi$, whence $\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, that is, $\varphi = \frac{\pi}{4} + 2\pi k$, $k \in Z$. ▲

By way of example, consider now a number of relations defining concrete subsets of the set of complex numbers C. Thus, the equality

$$|z| = 5 \tag{2}$$

is satisfied by the points of the complex plane equidistant from the origin, that is, equality (2) defines a circle of radius 5 centred at the

origin. In the complex plane, the double inequality $1 \leqslant |z| \leqslant 2$ specifies the set of points lying inside and on the boundary of the annulus formed by two concentric circles with centre at the origin O (Fig. 63). To the equality

$$\arg z = \frac{\pi}{4}$$

there corresponds in the complex plane the set of points lying on the ray emanating from the origin at an angle of 45° to the positive direction of the real axis (that is, of the axis of abscissas). The double inequality

$$\frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{3}$$

defines in the complex plane the set of points lying inside the angular sector (and on its boundaries) shown in Fig. 64.

Exercises

1. Find the arguments of the complex numbers:

(a) $z = i$; (b) $z = 1 - i$; (c) $z = -1$;
(d) $z = 7$; (e) $z = -5i$; (f) $z = 1 - 2i$.

2. What set of points in the complex plane is given by the condition: (a) the argument of the number z is equal to zero; (b) the argument is equal to $\frac{\pi}{2}$; (c) the argument is equal to $\frac{5\pi}{2}$; (d) the argument φ satisfies the inequalities $2\pi < \varphi < 3\pi$; (e) the argument φ satisfies the inequalities $0 \leqslant \varphi \leqslant 2\pi$?

3. What set of points in the complex plane is given by the condition $\arg z = (2n + 1)\pi$, $n \in \boldsymbol{Z}$?

4. What set of points in the complex plane is given by the condition: (a) $|z| \leqslant 1$; (b) $|z| = 7$; (c) $|z| > 3$; (d) $\sin |z| > 0$; (e) $\ln |z| < 1$?

Chapter 5

FUNCTIONS. SEQUENCES. LIMITS

5.1. Functions

1. The Concept of Function. The concept of function is one of the fundamental mathematical concepts. It is the most striking incarnation of the materialistic nature of mathematics, of its close relation with various phenomena of the reality. The concept of function involves establishing a relation between the members of two sets. For example, let A be the set of regular polygons with side equal to 3 cm, and B the set of perimeters of these polygons. It is clear that each of the given polygons can be associated with a definite number, that is, with its perimeter. Thus, the triangle from the set A is associated with the number 9, the square with the number 12, the pentagon with the number 15, and so forth. In this case we say that between the members of the set A and those of the set B a functional relation is established which is expressed by the following rule (law): to each regular polygon there corresponds a member—its perimeter.

If, according to a rule f, to every member x of the set A there corresponds the only member y from the set B, then we say that a function $f(x)$ is defined on the set A and write: $y = f(x)$, $x \in A$. In this case, x is called the argument, and y the value of the function. The set A is called the domain of definition of the function, while the set B the range of the function.

The following designation is often used for the function f: $A \to B$. Along with the letter f, other letters are also used: u, v, g, φ, ψ, . . ., F, Φ, Ψ, . . ., etc.

2. Numerical Functions. Ways of Representing Functions. Suppose we are given a function f: $A \to B$. If the numbers of the sets A and B are real numbers, that is, $A \subset \boldsymbol{R}$ and $B \subset \boldsymbol{R}$, then the function f is called a numerical function. As a rule, in the present course we shall deal with numerical functions, therefore, for brevity, we shall call them simply functions.

A function is said to be defined if we are given the sets A and B and the rule f according to which for an arbitrary $x \in A$ we can find (compute) the corresponding number $y \in B$. Frequently, this rule is given by a formula, for example:

$$y = 2x + 5$$

or

$$y = \frac{x^2+7}{2x-3}$$

and so on. This way of representing a function is called analytical. Note that one and the same formula can specify different functions depending on how the set A is indicated. Thus,

$$y = x,\ x \in \boldsymbol{R}, \quad \text{and} \quad y = x,\ x \in \boldsymbol{N},$$

are different functions: the first is a linear function, the second a sequence of natural numbers.

If a function represented by the formula $y = f(x)$ is defined on the set of all values of the variable x for which the expression $f(x)$ has sense, then the domain of its definition is usually not indicated. For instance, if the numerical function $f(x) = x^2 - 3x + 7$ is defined for all $x \in \boldsymbol{R}$, then it is usually given by a formula without indicating the domain of its definition; however, if the function $f(x) = x^2 - 3x + 7$ is specified on a certain subset of the set $\boldsymbol{R}$, then this should be necessarily stipulated. If a function is given by the formula $y = \frac{3}{x-2}$ without indicating the domain of its definition, then it is assumed that the domain of definition of this function is the set of all real numbers except 2 (for $x = 2$ the expression $\frac{3}{x-2}$ has no sense).

Sometimes, a function is specified by more than one formula to represent it on different numerical intervals. This can be exemplified by the formula

$$f(x) = \begin{cases} x+2, & x > 0, \\ x^3, & x \leqslant 0. \end{cases} \tag{1}$$

If it is difficult (or impossible) to write the formula by which each $x \in A$ is associated with $y \in B$, then we use a verbal description of the method representing a function. As an example, consider the following function: to each real number x there corresponds the largest integer not exceeding x. This function is called the integral part of x and is designated by $[x]$.

In practice, we often use the graphical method of representing a function. This method turns out to be convenient when it is rather difficult to represent a function analytically. Besides, nowadays many instruments and devices are used for studying various real processes. With the help of these instruments various curves are obtained by which we can judge about the character of the function under study. Oscillograms, electrocardiograms, etc. are examples of graphical description of functions. We can watch graphs of functions on visual displays of computers.

In mathematics, we often use the graphical method of representing a function even if the formula defining this function is known. A numerical function $f(x)$, $x \in A$ is completely defined by indicating the set of points $M(x, f(x))$, $x \in A$ on the number (coordinate) plane. This set of points is called the graph of the given function f.

To represent a function graphically means to represent its graph. Figure 65 shows the graph of function (1), where the arrow means that the point M (0, 2) does not belong to the graph. The graph of the function $y = [x]$ represented in Fig. 66 creates an easily perceptible and well memorizable "visual image" of this function.

Functions can be represented by tables (tabular method). In this case, we are given a table containing a certain number of values of

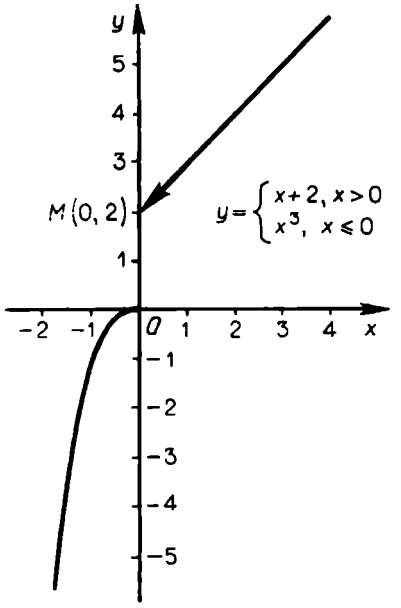

Fig. 65

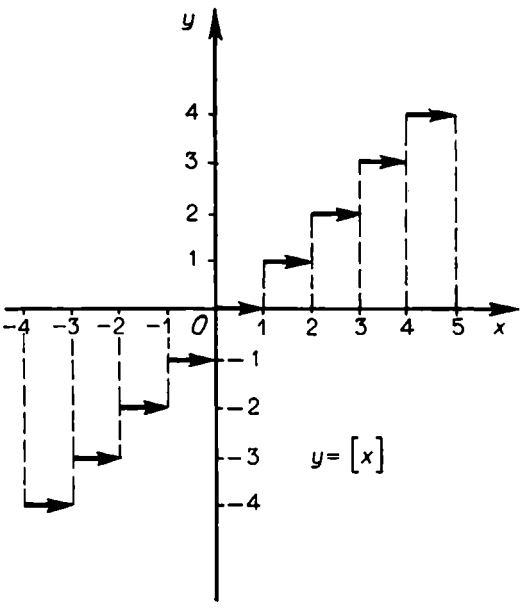

Fig. 66

the argument and the corresponding values of the function. These values can be obtained from experiments or with the help of certain mathematical calculations. For example, all of us use the tables of values of elementary functions (tables of square and cube roots, logarithmic tables, tables of trigonometric functions, etc.). Meteorologists compile precipitation tables for various areas ("points") of the Earth surface. In this case, these points play the role of the "values of the argument" and the amount of precipitation the role of the "values of the function".

Recently, functions are widely specified with the help of computer programs. When carrying out computer-aided calculations, operators put the needed values of arguments or the required values of functions in the computer storage in the ready-for-use form.

3. Boundedness, Monotonicity, Evenness and Periodicity of Functions.

Definition 1. A function

$$y = f(x), \quad x \in A, \tag{1}$$

is said to be bounded if there is a number M such that for any $x \in A$ the following inequality is satisfied:

$$|f(x)| \leqslant M. \tag{2}$$

Otherwise, the function is called unbounded. Geometrical condition (2) means that the graph of bounded function (1) is located on the coordinate plane in the horizontal band $-M \leqslant y \leqslant M$.

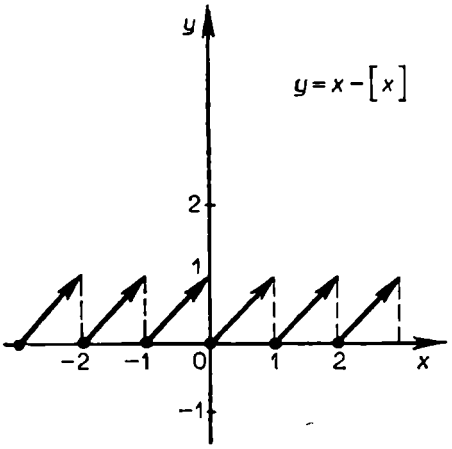

Fig. 67

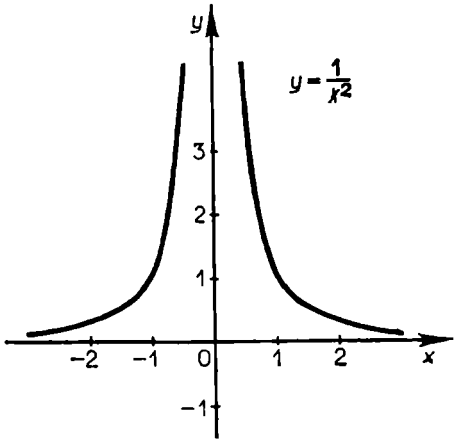

Fig. 68

Figure 67 shows the graph of the function $y = x - [x]$, $x \in \boldsymbol{R}$, entirely situated in the horizontal band $0 \leqslant y \leqslant 1$. Therefore the given function is bounded. Figure 68 shows the graph of the function $y = \frac{1}{x^2}$ which is an unbounded function. A couple of unbounded functions were considered in the preceding subsection (see Figs. 65 and 66).

Definition 2. A function $f(x)$, $x \in A$, is said to be increasing if for any x_1 and x_2 from A such that $x_1 < x_2$ the inequality

$$f(x_1) \leqslant f(x_2)$$

is satisfied. If $f(x_1) < f(x_2)$, then the function is said to be strictly increasing.

Definition 3. A function $f(x)$, $x \in A$, is said to be decreasing if for any x_1 and x_2 from A such that $x_1 < x_2$ the inequality

$$f(x_1) \geqslant f(x_2)$$

is satisfied. If $f(x_1) > f(x_2)$, then the function is said to be strictly decreasing.

Strictly increasing, increasing, strictly decreasing, and decreasing functions are called monotonic functions. Strictly increasing and strictly decreasing functions are termed strictly monotonic functions.

Example. Show that the function $f(x) = x^3$, $x \in \boldsymbol{R}$, is strictly increasing.

△ Let $x_1 < x_2$. Then $f(x_2) - f(x_1) = x_2^3 - x_1^3 = (x_2 - x_1) \times (x_2^2 + x_2x_1 + x_1^2)$. Since $x_2^2 + x_2x_1 + x_1^2 > 0$ for any x_1 and $x_2 \neq x_1$, we have $f(x_2) - f(x_1) > 0$, that is, $f(x_1) < f(x_2)$. The graph of the function $y = x^3$ is shown in Fig. 69. ▲

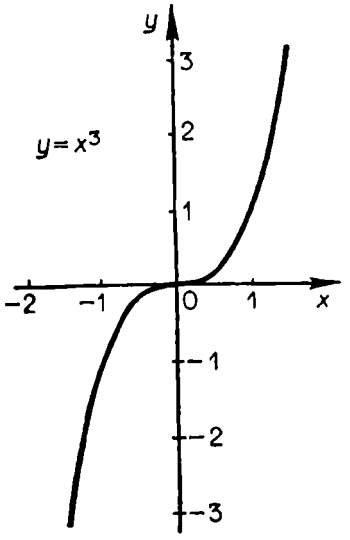

Fig. 69

Similarly, we can prove that the function $y = [x]$, $x \in \boldsymbol{R}$ is increasing (Fig. 66), the function $y = -x^3$, $x \in \boldsymbol{R}$, strictly decreasing (Fig. 70, *a*), the function $f(x) = -[x]$, $x \in \boldsymbol{R}$, decreasing (Fig. 70, *b*), while the function $y = x - [x]$, $x \in \boldsymbol{R}$, possesses no monotonicity property (Fig. 67).

Definition 4. A function $f(x)$, $x \in A$, is said to be even (odd) if for any x from the set A the equality $f(-x) = f(x)$ ($f(-x) = -f(x)$) is satisfied.

The function $y = x^3$, $x \in \boldsymbol{R}$, whose graph is shown in Fig. 69, is odd, the function $y = \frac{1}{x^2}$ (see Fig. 68) is even, while the functions whose graphs are shown in Figs 65, 66, 67, and 70, *b* possess neither evenness nor oddness properties. Other examples of even functions are the following functions defined throughout the number line:

$$f(x) = x^4, \quad f(x) = -|x|, \quad f(x) = |x| - 1.$$

The function $f(x) = \sqrt{4 - x^2}$, defined on the closed interval $[-2, 2]$, and also the function $f(x) = \frac{1}{x^4}$, $x \in \boldsymbol{R}$, $x \neq 0$, are even.

The following functions:

$$f(x) = x, \quad f(x) = -x^3, \quad f(x) = x|x|,$$

defined throughout the number line, are odd. The function $f(x) = x\sqrt{9 - x^2}$, defined on the closed interval $[-3, 3]$ and the function $y = \frac{1}{x}$, $x \in \boldsymbol{R}$, $x \neq 0$ are also odd.

As is seen from the definitions of an even and odd functions, together with the number x, the number $-x$ also belongs to the domains of definition of such functions.

Since the graph of an even function is symmetrical about the axis of ordinates (see Fig. 68), and the graph of an odd function is symmetrical about the origin (see Figs. 69 and 70, *a*), it suffices to construct their graphs only for $x \geqslant 0$ and then reflect the obtained lines symmetrically about the axis of ordinates or the origin.

Definition 5. A function $y = f(x)$, $x \in A$, is called periodic if there exists a number $T \neq 0$ such that for any x from the set A the equality $f(x - T) = f(x) = f(x + T)$ is satisfied. In this case,

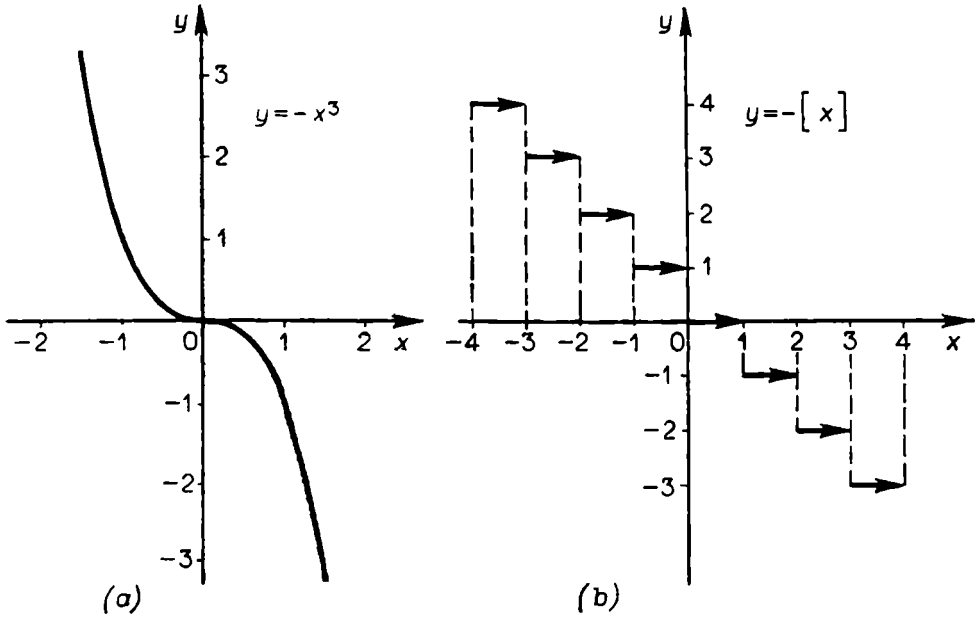

Fig. 70

the number T is called the period of the function f. Among the functions whose graphs are shown in Figs. 65-70, only the function $y = x - [x]$, $x \in \boldsymbol{R}$, possesses the periodicity property. Its period T is equal to unity. The rest of the functions are said to be nonperiodic.

Exercises

1. Evaluate the functions $f(2)$, $f(-1)$, $f\left(\frac{1}{3}\right)$, $f(-2)$ if:

(a) $f(x) = x^2$; (b) $f(x) = \frac{x^2-1}{x-4}$; (c) $f(x) = x^2 - 3x + 2$;

(d) $f(x) = \frac{6}{x-1}$.

2. Find the domain of definition of the function given by the formula:

(a) $y = x^3 - x$; (b) $y = |x| + 1$; (c) $y = \frac{1}{x+2}$;

(d) $y = \frac{x^3-1}{x-1}$; (e) $y = \sqrt[3]{x}$.

3. Sketch the graphs of the following functions:

(a) $y=x^2-4$; (b) $y=5x-3$; (c) $y=x^2-2x+3$;

(d) $y=|x-2|$; (e) $y=\frac{1}{x-3}$; (f) $y=\begin{cases} 1 \text{ if } x>0 \\ 0 \text{ if } x=0, \\ -1 \text{ if } x<0. \end{cases}$

4. Which of the lines shown in Fig. 71 are graphs of functions?

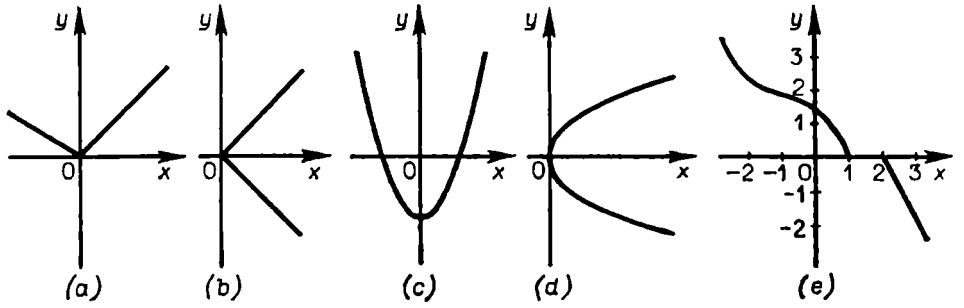

Fig. 71

5. Determine which of the given functions are even, and which are odd:

(a) $f(x)=x^3-5$; (b) $f(x)=x+7$; (c) $f(x)=|3-x|$;

(d) $f(x)=x^4+x^2$; (e) $f(x)=-x^2$.

6. Is the given function $f(x)$ periodic:

(a) $f(x)=8$; (b) $f(x)=\begin{cases} 1 \text{ if } x \text{ is a rational number,} \\ 0 \text{ if } x \text{ is an irrational number?} \end{cases}$

7. Which of the functions is increasing for $x>0$:

(a) $y=2x-4$; (b) $y=x^2-9$; (c) $y=\frac{8}{x}$?

8. Which of the functions is decreasing for $x<0$:

(a) $y=-x+2$; (b) $y=-x^2+4$; (c) $y=\frac{3}{x}$?

9. Solve the given equations graphically:

(a) $x^2=6x+5$; (b) $x^2=x$; (c) $|x-1|=4$; (d) $\frac{3}{x}=x$.

10. Solve the given inequalities graphically:

(a) $3x-1>4x+1$; (b) $x^2>x$; (c) $x^2-7x+15>3$.

5.2. The Inverse of a Function. Simplest Elementary Functions

1. The Inverse of a Given Function. Let a function $y = f(x)$, $x \in A$ be given, and let B be the set of values of the given function. The function f associates a unique member y of the set B with a member x of the set A. In addition, let us demand that the function f sets in one-to-one correspondence between distinct members x_1 and x_2 of the set A and distinct members y_1 and y_2 of the set B. Then we can consider a function which associates any element y of the set B with a unique member x of the set A such that $f(x) = y$. This function is called the inverse of the given function f and is denoted by f^{-1} or $x = f^{-1}(y)$, $y \in B$. For the function f^{-1} the set A is the range (or the set of values), and the set B the domain of definition.

Example 1. Find the inverses of the functions $y = x^2$, $x \in [0, 1]$ and $y = x^2$, $x \in [-1, 1]$, provided that they exist.

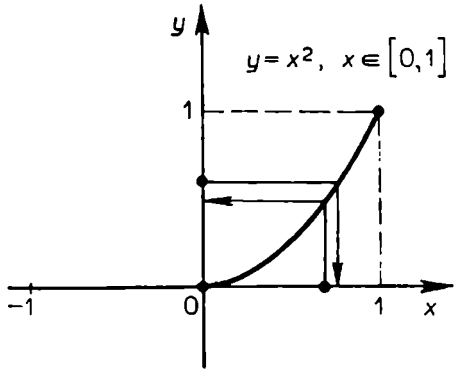

Fig. 72

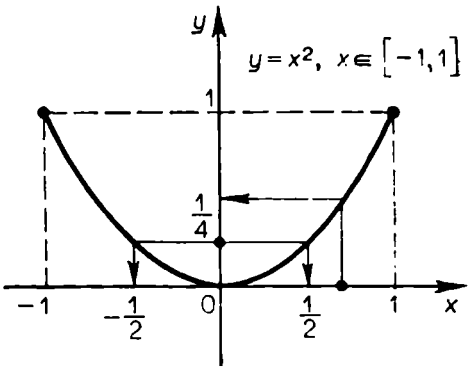

Fig. 73

△ For the function $y = x^2$, $x \in [0, 1]$, the function $x = \sqrt{y}$, $y \in [0, 1]$, is its inverse (Fig. 72).

The function $y = x^2$, $x \in [-1, 1]$ has no inverse since one and the same y corresponds to distinct x_1 and x_2. For example, to the numbers $x_1 = \frac{1}{2}$ and $x_2 = -\frac{1}{2}$ one and the same number $y = \frac{1}{4}$ corresponds (Fig. 73). ▲

A function having an inverse is said to be reversible. Obviously, in order to specify the function f^{-1}, that is, the inverse of the given function f, it suffices to solve the equation $y = f(x)$ with respect to x (if it is possible), expressing thereby the variable x in terms of y: $x = f^{-1}(y)$. In turn, the function f is the inverse of the function f^{-1}. The functions f and f^{-1} are said to be mutually inverse.

Example 2. Find the inverse of the function

$$y = 2x - 5, \quad x \in R. \tag{1}$$

△ The set of values of function (1) is the set of real numbers R. Expressing x in terms of y from the equation $y = 2x - 5$, we get

$$x = \frac{y+5}{2}, \quad y \in R. \ \blacktriangle \tag{2}$$

Mutually inverse functions (1) and (2) define on the coordinate plane one and the same straight line l which is the graph of both of these functions (Fig. 74). The graph of mutually inverse functions $y = f(x)$ and $x = f^{-1}(y)$ is one and the same curve l in the coordinate plane, the values of arguments for the function $f^{-1}(y)$ being

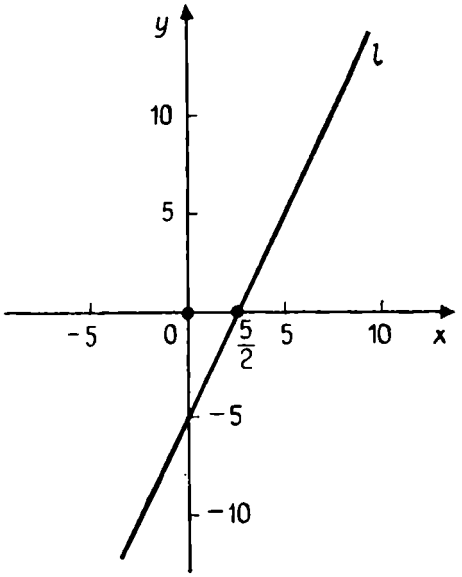

Fig. 74

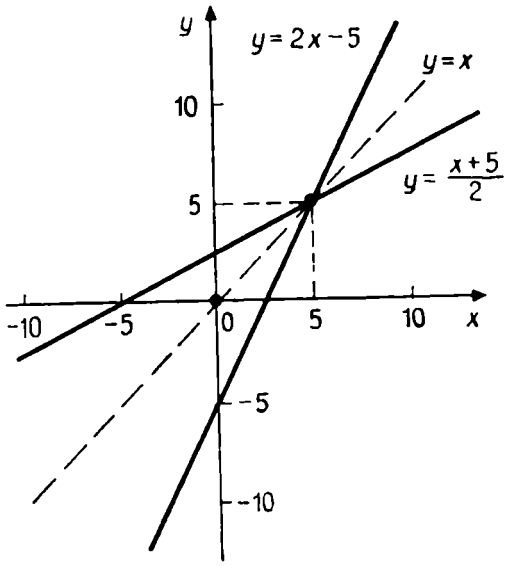

Fig. 75

laid off on the axis of ordinates which is inconvenient. Therefore it is customary to use for the inverse function f^{-1} the same designation as for the function f, that is, $y = f^{-1}(x)$ and to lay off the values of the arguments of the inverse function on the axis of abscissas. In this case, the graphs of the functions $y = f(x)$ and $y = f^{-1}(x)$ will be symmetric to each other with respect to the straight line $y = x$. The graphs of mutually inverse functions $y = 2x - 5$, $x \in R$, and $y = \frac{x+5}{2}$, $x \in R$ (obtained from formula (2) by changing the designation) considered in Example 2 are shown in Fig. 75.

2. Simplest Elementary Functions. In the present subsection, we are going to consider such simplest elementary functions as power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

For any real number α the function

$$y = x^{\alpha}, \quad x \in \boldsymbol{R}_{+},$$

is called the power function with exponent α. If $\alpha = 0$, then $x^{\alpha} = 1$ for any $x > 0$. If $\alpha \neq 0$, then the set of values of the power function is the interval $(0, +\infty)$. Indeed, the function takes on each value $y_0 > 0$ at the point $x_0 = y_0^{1/\alpha}$. Thus, the power function is an unbounded function. A power function with a positive exponent is

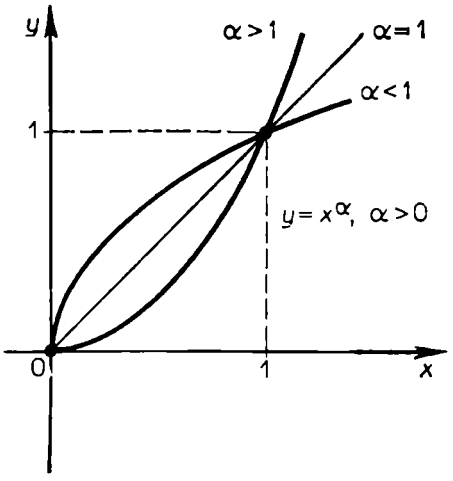

Fig. 76

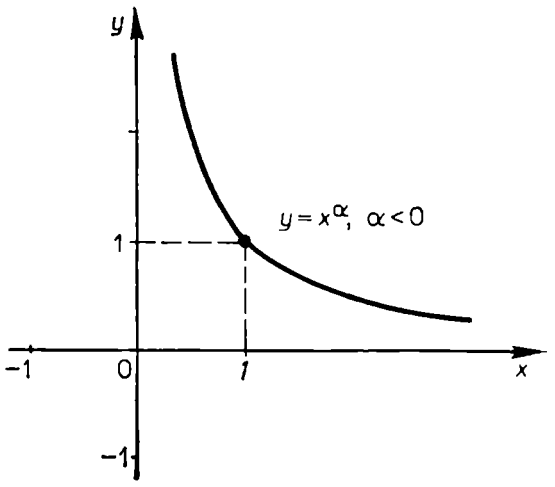

Fig. 77

increasing, while with a negative exponent it is decreasing. Figures 76 and 77 represent the graphs of the power function for $\alpha > 1$, $\alpha = 1$, $\alpha \in (0, 1)$ and $\alpha < 0$. For certain α the power x^{α} is defined not only for $x > 0$. For instance, if $\alpha = \frac{m}{n}$, where n is an odd natural number, and m any natural number, then the power function $y = x^{\alpha}$ is defined for any $x \in \boldsymbol{R}$. If $\alpha = -n$, where n is a natural number, then the function x^{α} is defined for any $x \in \boldsymbol{R}$, $x \neq 0$. This function is even if n is even and it is odd if n is odd.

For any $a > 0$, $a \neq 1$ the function

$$y = a^{x}, \quad x \in \boldsymbol{R},$$

is called exponential. The set of values of the exponential function is the set $\boldsymbol{R}_{+}$ of all positive real numbers. Using the properties of powers, it is easy to prove that if $a > 1$, then the exponential function is increasing, and if $0 < a < 1$, then it is decreasing. In both cases the function is unbounded.

For any $a > 0$, $a \neq 1$ the inverse of the exponential function

$$y = \log_a x, \quad x \in \boldsymbol{R}_{+},$$

is called the logarithmic function. The set of values of the logarithmic function is the set R of all real numbers. If $a > 1$, then the logarithmic function is increasing, and if $0 < a < 1$, then it is decreasing.

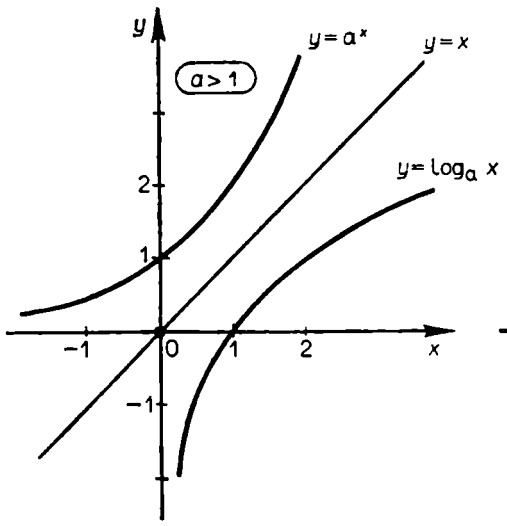

Fig. 78

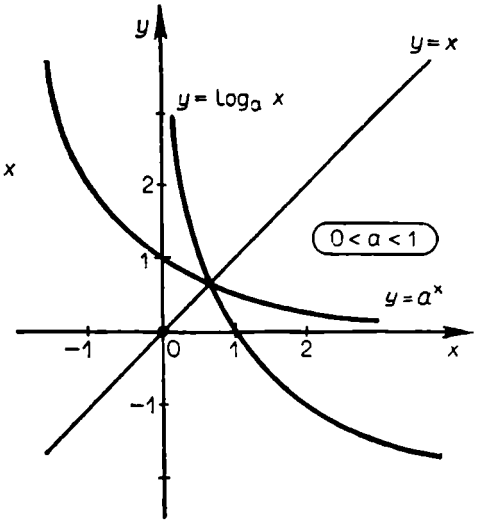

Fig. 79

The graphs of mutually inverse exponential and logarithmic functions are symmetric to each other with respect to the straight line $y = x$ (Figs. 78 and 79).

The trigonometric functions

$$y = \sin x, \quad x \in R, \quad y = \cos x, \quad x \in R,$$

are called the sine and cosine. The graphs of these functions are shown in Figs. 80 and 81. They entirely lie in the band $-1 \leqslant y \leqslant 1$

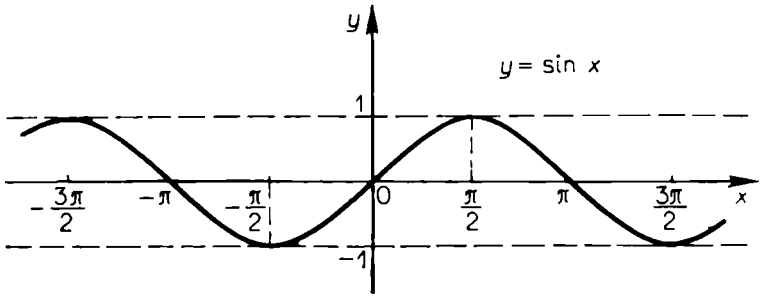

Fig. 80

since these functions are bounded. The function $\cos x$ is even, and the function $\sin x$ is odd. Both of them are periodic with period 2π:

$$\sin (x - 2\pi) = \sin x = \sin (x + 2\pi),$$

$$\cos (x - 2\pi) = \cos x = \cos (x + 2\pi)$$

for any $x \in R$. Apart from the sine and cosine we consider the functions

$$\tan x = \frac{\sin x}{\cos x}, \quad x \in R, \quad x \neq \pm \frac{\pi}{2} + \pi k, \quad k \in Z,$$

$$\cot x = \frac{\cos x}{\sin x}, \quad x \in R, \quad x \neq \pi k, \quad k \in Z,$$

called the tangent and cotangent. The graphs of these functions are shown in Figs. 82 and 83; $\tan x$ and $\cot x$ are odd, unbounded, periodic functions with period π.

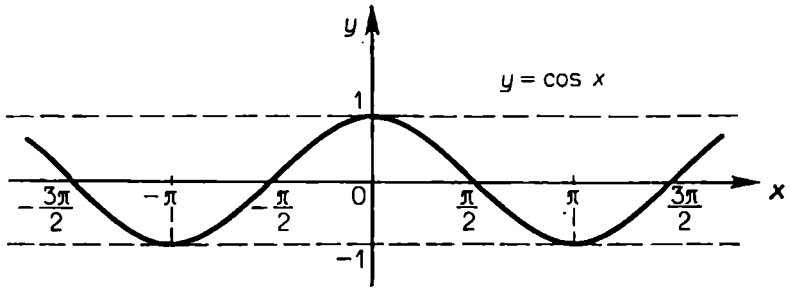

Fig. 81

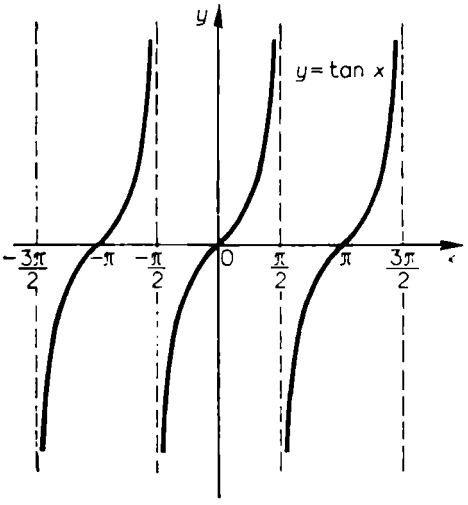

Fig. 82

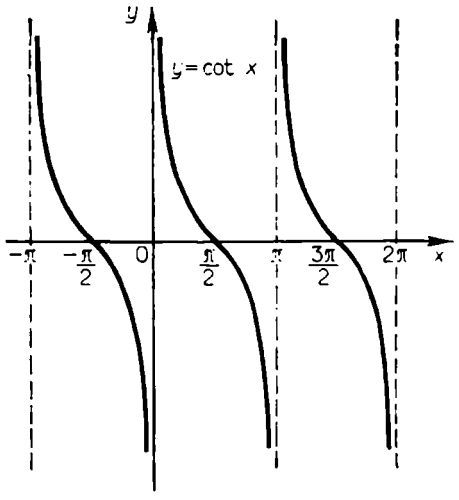

Fig. 83

The function $y = \sin x$ strictly increases on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that different values of the function y correspond to different values of the argument x. The inverse of the function $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is called the arc sine and is denoted as $y = \arcsin x$. It is defined on the closed interval $[-1, 1]$. The set of values of the arc sine is the closed interval $\left[-\frac{\pi}{2},\right.$

$\frac{\pi}{2}\Big]$. The graph of the function $y=\arcsin x$, $x\in[-1, 1]$ is symmetric to the graph of the function $y=\sin x$, $x\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with respect to the straight line $y=x$ (Fig. 84). The arc sine is an increasing odd function.

The function $y=\cos x$ strictly decreases on the interval $[0, \pi]$ so that different values of the function y correspond to different values

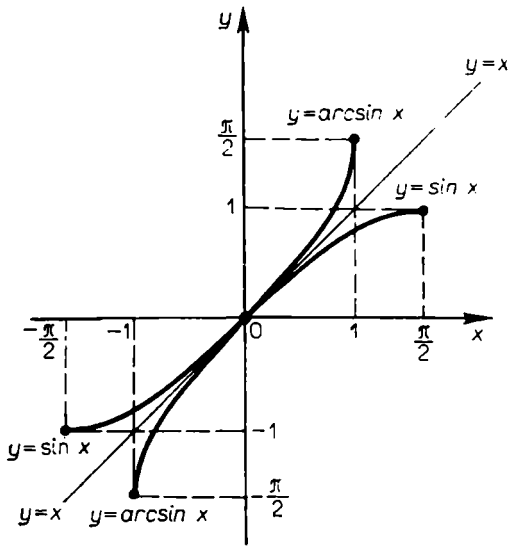

Fig. 84

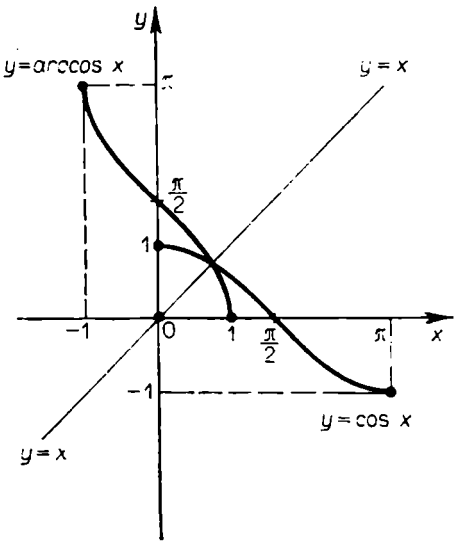

Fig. 85

of the argument x. Then, it is possible to introduce the inverse of the function $y=\cos x$, $x\in[0, \pi]$. This function is called arc cosine and is denoted by $y=\arccos x$. It is defined on the interval $[-1, 1]$. The set of values of the arc cosine is the interval $[0, \pi]$. The graph of the function $y=\arccos x$, $x\in[-1, 1]$ is symmetric to the graph of the function $y=\cos x$, $x\in[0, \pi]$, with respect to the straight line $y=x$ (Fig. 85). The arc cosine is a decreasing function. It possesses neither the evenness nor oddness property.

The function $y=\tan x$, $x\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is strictly increasing, and different values of the function y correspond to different values of the argument x. The inverse of the function $y=\tan x$, $x\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is called the arc tangent and is denoted by $y=\arctan x$. It is defined for all $x\in\boldsymbol{R}$. The set of values of the arc tangent is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The graph of the function $y=\arctan x$, $x\in\boldsymbol{R}$. is symmetric to the graph of the func-

tion $y=\tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with respect to the straight line $y=x$ (Fig. 86). The arc tangent is an increasing odd function.

The function $y = \cot x$, $x \in (0, \pi)$, strictly decreases, and different values of the function y correspond to different values of the argument x. The inverse of the function $y = \cot x$, $x \in (0, \pi)$, is called the arc cotangent and is denoted by $y = \operatorname{arccot} x$. It is defined for all $x \in \boldsymbol{R}$. The set of values of the arc cotangent is the interval $(0, \pi)$.

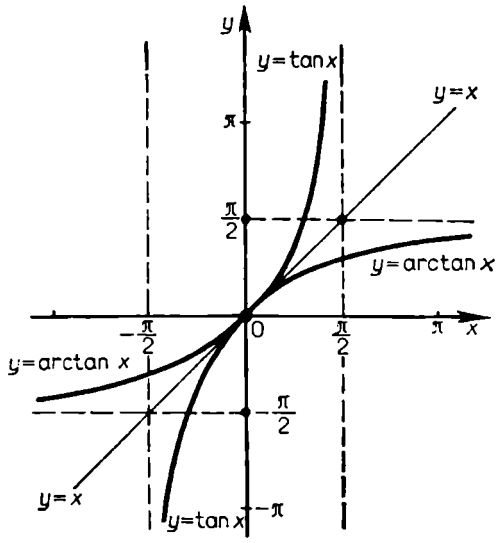

Fig. 86

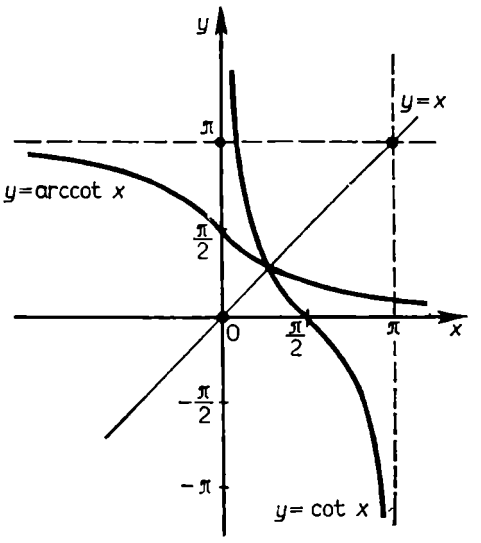

Fig. 87

The graph of the function $y = \operatorname{arccot} x$, $x \in \boldsymbol{R}$, is symmetric to the graph of the function $y = \cot x$, $x \in (0, \pi)$, with respect to the straight line $y = x$ (Fig. 87). The arc cotangent is a decreasing function. This function possesses neither the evenness nor oddness property.

The considered inverse trigonometric functions $\arcsin x$, $\arccos x$, $\arctan x$, and $\operatorname{arccot} x$ are monotonic, nonperiodic functions bounded in their domains of definition.

The constant function $y = C$, $x \in \boldsymbol{R}$ also belongs to the simplest elementary function, the only number C being its set of values. The graph of this function is a straight line parallel to the axis of abscissas and located at a distance $|C|$ from it. This straight line lies above the axis of abscissas if $C > 0$, below the axis of abscissas if $C < 0$, and coincides with the axis of abscissas if $C = 0$. The constant function is a bounded even periodic function; in this case, any number $T \in \boldsymbol{R}$ may serve as period.

Let us, for obviousness, tabulate the simplest elementary functions considered above (Table 1).

Table 1

Function	Domain A	Range B	Boundedness	Evenness	Oddness	Monotonicity	Periodicity
I. Constant function 1. $y=C$, $C\in R$	$A=R$	$B=\{C\}$	Yes	Yes	No	No	Yes T—any
II. Power function 2. $y=x^{\alpha}$, $\alpha\in R$, $\alpha\neq 0$	$A=R_+$	$B=R_+$	No	No	No	Yes	No
III. Exponential function 3. $y=a^x$, $a>0$, $a\neq 1$	$A=R$	$B=R_+$	No	No	No	Yes	No
IV. Logarithmic function 4. $y=\log_a x$, $a>0$, $a\neq 1$	$A=R_+$	$B=R$	No	No	No	Yes	No
V. Trigonometric functions 5. $y=\sin x$	$A=R$	$B=[-1, 1]$	Yes	No	Yes	No	Yes T=2π
6. $y=\cos x$	$A=R$	$B=[-1, 1]$	Yes	Yes	No	No	Yes T=2π
7. $y=\tan x$	$A=R$, $x\neq\frac{\pi}{2}+\pi k$	$B=R$	No	No	Yes	No	Yes T=π
8. $y=\cot x$	$A=R$, $x\neq\pi k$	$B=R$	No	No	Yes	No	Yes T=π

Table 1 (continued)

Function	Domain A	Range B	Boundedness	Evenness	Oddness	Monotonicity	Periodicity
VI. Inverse trigonometric functions							
9. $y=\arcsin x$	$A=[-1, 1]$	$B=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Yes	No	Yes	Yes	No
10. $y=\arccos x$	$A=[-1, 1]$	$B=[0, \pi]$	Yes	No	No	Yes	No
11. $y=\arctan x$	$A=R$	$B=\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Yes	No	Yes	Yes	No
12. $y=\operatorname{arccot} x$	$A=R$	$B=(0, \pi)$	Yes	No	No	Yes	No

Exercises

1. Find the inverse of the given function:

(a) $y=3x-2$; (b) $y=\frac{5}{x-1}$; (c) $y=\frac{2x-5}{4-x}$;

(d) $y=|x-2|$; (e) $y=x^2$.

2. Sketch on one sheet of paper the graphs of the functions $y=3^x$, $y=2^x$, and $y=\left(\frac{3}{2}\right)^x$. Indicate the similarity and difference among the graphs of these functions.

3. Do the same as in Exercise 2 for the functions $y=\left(\frac{1}{3}\right)^x$, $y=\left(\frac{1}{2}\right)^x$, $y=\left(\frac{2}{3}\right)^x$.

4. Find the domain of definition and the set of values of the following functions: (a) $y=2^{|x|}$; (b) $y=-2^x$; (c) $y=|3^x-3|$. Sketch the graphs of these functions.

5. Sketch on one sheet of paper the graphs of the functions $y=\log_3 x$, $y=\log_{3.5} x$, $y=\log_5 x$. Indicate the similarity and difference between the graphs of these functions.

6. Do the same as in Exercise 6 for the functions $y=\log_{\frac{1}{3}} x$, $y=\log_{0.5} x$, $y=\log_{\frac{1}{4}} x$.

7. Find the domain of definition and the set of values for the following functions: (a) $y=\log_2|x|$; (b) $y=\log_{0.5}|x|$; (c) $y=|\log_2 x|$; (d) $y=|\log_{\frac{1}{2}} x|$; (e) $y=\log_2(-x)$; (f) $y=|\log_{\frac{1}{2}}(-x)|$. Construct the graphs of these functions.

8. Sketch on one sheet of paper the graphs of the functions $y=x$, $y=x^2$, $y=x^{1/2}$, $y=x^{2/3}$, $y=x^{3/2}$. Indicate the similarity and difference between the graphs of these functions.

9. Sketch on one sheet of paper the graphs of the following pairs of functions:

(a) $y=\sin x$ and $y=-2\sin x$; (b) $y=\sin x$ and $y=\sin\left(-\frac{x}{2}\right)$;

(c) $y=\cos x$ and $y=2\cos x$; (d) $y=\cos x$ and $y=\cos 2x$.

10. Evaluate the function $y=\arccos x-\arctan 2x$ for $x=0$, $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$.

11. Evaluate the function $y=\operatorname{arccot} 2x-\arcsin x$ for $x=-\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$.

12. Graph the following functions:

(a) $y=2\arcsin x$; (b) $y=\arcsin 2x$;

(c) $y=\frac{1}{2}\arccos x$; (d) $y=\arccos\frac{x}{2}$.

5.3. The Composite Function. The Class of Elementary Functions. Polynomials. Rational Functions

1. The Composite Function. Let there be given two functions $y=g(x)$, $x\in X$, and $z=\varphi(y)$, $y\in Y$, the domain of definition of the function φ containing the set of values of the function g. In this case, the function $z=\varphi(g(x))$ composed of the functions g and φ is called the composite function or the superposition of the functions g and φ. For instance, the function $z=\ln(1+x^3)$ is a composite function composed of two simpler functions $y=1+x^3$ and $z=\ln y$. Composite functions which are superpositions of more than two functions are considered in a similar way. For instance, the function $z=\sin(1+\sqrt[3]{x})$ can be regarded as a superposition of the following functions: $z=\sin v$, $v=1+y$ and $y=\sqrt[3]{x}$.

Example. Compose $g(\varphi(x))$ and $\varphi(g(x))$ for the functions $g(x) = x^2 + \sqrt[3]{x}$ and $\varphi(x) = \ln x + x^5 + 1$.

△ In order to compose the composite function $g(\varphi(x))$, we set $z = g(y)$, $y = \varphi(x)$. Then $z = y^2 + \sqrt{y} = (\ln x + x^5 + 1)^2 + \sqrt{\ln x + x^5 + 1}$. Thus, we have found that

$$z = g(\varphi(x)) = (\ln x + x^5 + 1)^2 + \sqrt[3]{\ln x + x^5 + 1}.$$

Similarly, setting $z = \varphi(y)$, $y = g(x)$, we find $z = \ln y + y^5 + 1 = \ln(x^2 + \sqrt[3]{x}) + (x^2 + \sqrt[3]{x})^5 + 1$. Thus,

$$z = \varphi(g(x)) = \ln(x^2 + \sqrt[3]{x}) + (x^2 + \sqrt[3]{x})^5 + 1.$$

In the usual fashion, it is assumed here that the obtained functions are defined on natural domains of definition. ▲

The considered example shows that the result of a superposition of two different functions depends on the order in which these functions follow, that is, generally speaking,

$$\varphi(g(x)) \neq g(\varphi(x)) \quad \text{if} \quad \varphi(x) \neq g(x).$$

Consider the superposition of mutually inverse functions f and f^{-1}. If A is the domain of definition of the function f, and B the set of its values, then we have the following equalities:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for any} \quad x \in A, \\ f(f^{-1}(y)) &= y \quad \text{for any} \quad y \in B, \end{aligned} \tag{1}$$

since a superposition of mutually inverse mappings is an identical mapping.

Consider the inverse trigonometrical functions $\arcsin x$ and $\arccos x$ defined on the interval $[-1, 1]$. On this interval

$$\begin{aligned} \sin(\arcsin x) &= x, \\ \cos(\arccos x) &= x. \end{aligned} \tag{2}$$

Using formula (2), we find:

$$\begin{aligned} \cos(\arcsin x) &= \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1 - x^2}, \\ \sin(\arccos x) &= \sqrt{1 - \cos^2(\arccos x)} = \sqrt{1 - x^2}. \end{aligned} \tag{3}$$

The radicals are taken with the plus sign, since on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (this is the set of values of the function $\arcsin x$) the cosine is nonnegative, and the sine is nonnegative on the interval $[0, \pi]$ (this is the set of values of the function $\arccos x$). Let us prove the equality

$$\arcsin x + \arccos x = \frac{\pi}{2}, \quad x \in [-1, 1]. \tag{4}$$

□ To prove formula (4), it suffices to show that for any $x \in [-1, 1]$ the equality

$$\sin(\arcsin x + \arccos x) = 1$$

is valid. Let us make use of the known trigonometrical equality $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$ and formulas (2) and (3). Then

$$\sin(\arcsin x + \arccos x) = \sin(\arcsin x) \cdot \cos(\arccos x)$$
$$+ \cos(\arcsin x) \cdot \sin(\arccos x) = x^2 + (\sqrt{1 - x^2})^2 = 1. \blacksquare$$

Similarly, we prove the equality

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}, \quad x \in \boldsymbol{R}. \tag{5}$$

Relations (4) and (5) between the inverse trigonometrical functions will be widely used.

2. Polynomials. A polynomial of degree n or an entire rational function is defined as a function of the form

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n, \quad x \in \boldsymbol{R}, \quad a_0 = 0, \tag{1}$$

where a_0, a_1, . . ., a_{n-1}, a_n are arbitrary fixed real numbers called the coefficients of the polynomial, and $n \in N$. Polynomials are customarily denoted by capital letters P, Q, . . ., etc., its degree n being indicated by a subscript. $P_3(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$, $P_2(x) = 5x^2 - 3x + 7$, $Q_1(x) = a_0 x + 4$, etc. are examples of polynomials. A polynomial of degree zero ($n = 0$) coincides with a constant function

$$P_0(x) = a_0, \quad x \in \boldsymbol{R},$$

where, as distinct from formula (1), the constant a_0 may be equal to zero. For a polynomial of general form (1) some coefficients (except for a_0) may be equal to zero. The polynomials $P_3(x) = a_0 x^3 + 7$, $P_2(x) = 2x^2 - 5x$, etc. are examples of such incomplete polynomials.

The polynomial of the first degree

$$P_1(x) = a_0 x + a_1$$

is called a linear function. The polynomial of the second degree

$$P_2(x) = a_0 x^2 + a_1 x + a_2$$

is conventionally called a quadratic trinomial.

Figures 88 and 89 show the graphs of the polynomials $y = x^2 - 2x - 3$ and $y = \frac{1}{3}x^3 - x^2 - 1$, respectively. Polynomials are

unbounded and nonperiodic functions. In the general case, they possess no properties of evenness (oddness) and monotonicity. Some polynomials (for instance, $P_2(x) = x^2$) turn out to be even functions, and some (as for instance, $P_3(x) = x^3$) monotonic functions.

3. Rational Functions. A function of the form

$$y = \frac{P_n(x)}{Q_m(x)} = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}$$

is called a rational function (rational fraction). The domain of definition of such a function is the entire number line, except for the

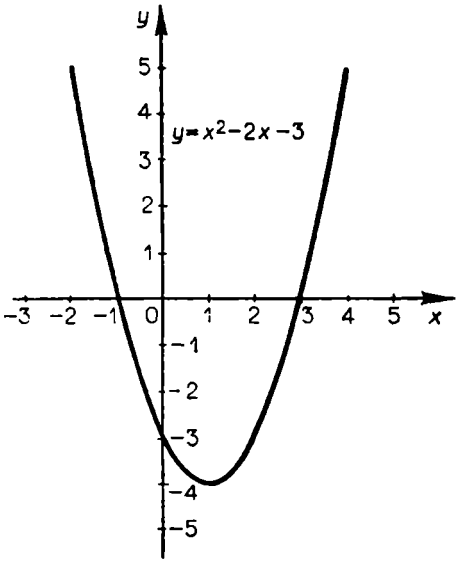

Fig. 88

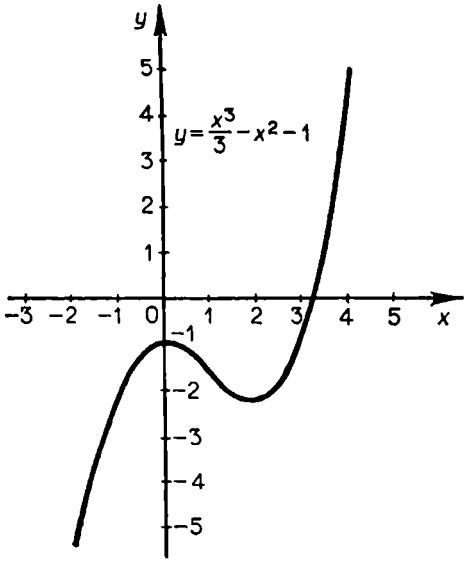

Fig. 89

points at which the denominator vanishes. The polynomial is, obviously, a particular case of the rational function. The examples of rational functions are: $y = \frac{5x^3 - 2x + 3}{x^2 + x - 7}$, $y = \frac{2x - 1}{x^2 - 2x - 3}$, etc. Figure 90 shows the graph of the rational function $y = \frac{2x - 1}{x^2 - 2x - 3}$.

A rational fraction is said to be proper if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Otherwise, a rational fraction is said to be improper. An improper rational fraction can always be represented as a sum of a polynomial and a proper rational fraction. To this end, it suffices to divide the numerator by the denominator. Consider, for example, the improper rational fraction

$$y = \frac{x^2 + 2}{x - 1}.$$

We divide the numerator by the denominator:

$$\begin{array}{r|l} x-1 & \begin{array}{l} x+1 \\ x^2+2 \\ -x^2-x \\ \hline \quad x+2 \\ \quad -x-1 \\ \hline \quad\quad 3 \end{array} \end{array}.$$

Hence we find that

$$y = \frac{x^2+2}{x-1} = x+1+\frac{3}{x-1}.$$

Thus, the improper rational fraction $\frac{x^2+2}{x-1}$ has been represented as a sum of the polynomial $x+1$ and the proper rational fraction $\frac{3}{x-1}$.

In a particular case, when the denominator vanishes nowhere, proper rational functions turn out to be bounded functions defined

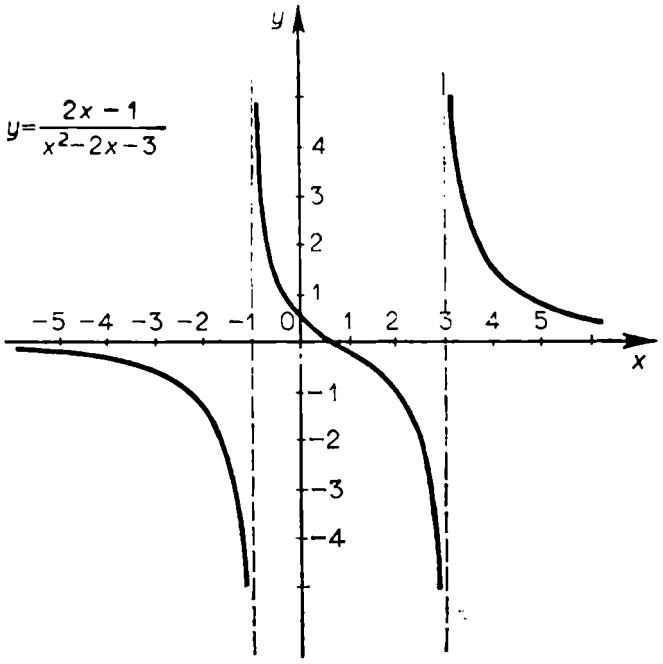

Fig. 90

through the number line. This can be exemplified by the function $y = \frac{x+1}{x^2+3}$, $x \in R$, whose graph is shown in Fig. 91.

4. **Elementary Functions.** A function defined by a formula involving only the following operations on the argument x: addition, subtraction, multiplication, division, and raising to a rational power,

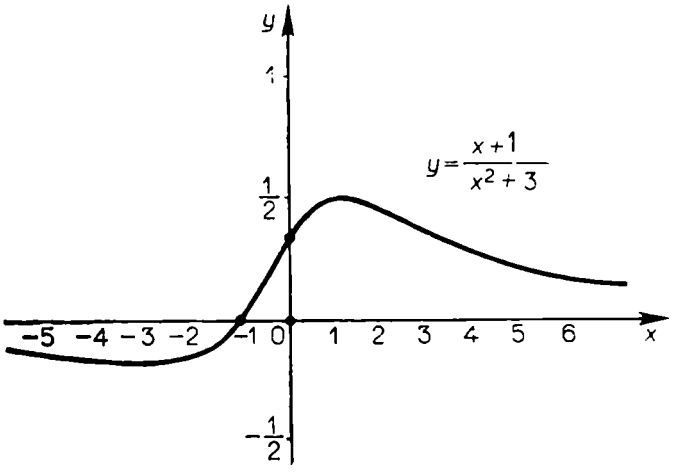

Fig. 91

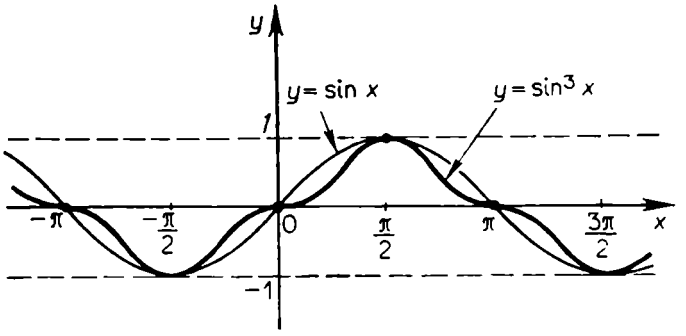

Fig. 92

is called algebraic. It is clear that polynomials and rational functions are algebraic functions. An algebraic function is said to be irrational if the formula defining this function contains radicals. For example,

$$y=\frac{5x^2-\sqrt{x}}{2x^4+\sqrt[5]{x^2}}$$

is an irrational function.

The functions a^x, $\log_a x$, $\sin x$, and other trigonometric functions are called transcendental functions. In other words, functions which are not algebraic are termed transcendental.

Functions obtained with the help of a finite number of arithmetical operations and a finite number of superpositions composed from simplest elementary functions play an important role in applications. Such functions are called elementary functions. The functions

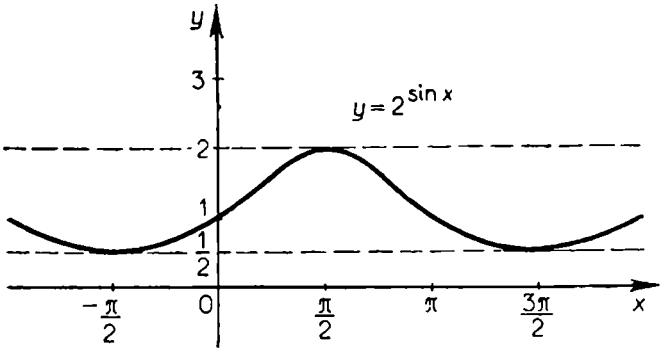

Fig. 93

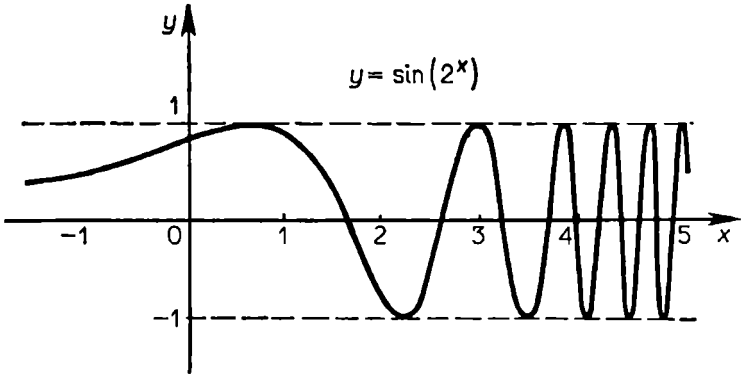

Fig. 94

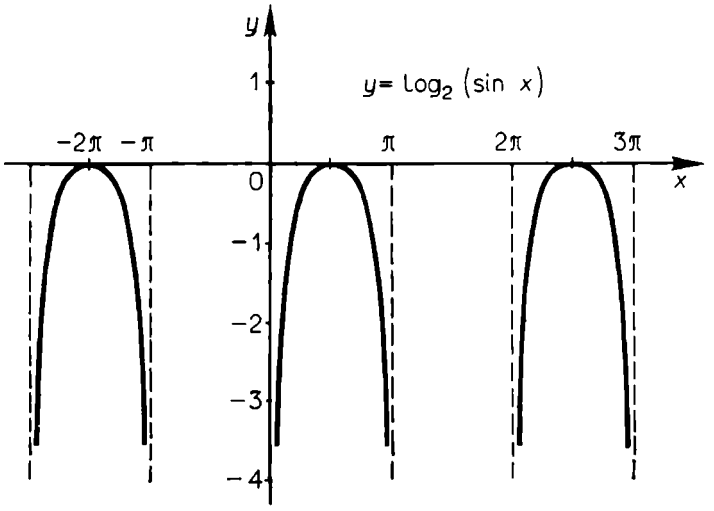

Fig. 95

$f_1(x) = x^2 + 5 \sin 3x$, $f_2(x) = \ln |\cos x|$, $f_3(x) = \sin(\ln x) + e^{\arcsin\sqrt{x}}$ are examples of elementary functions.

The class of elementary functions consists of an infinite number of functions possessing a variety of properties. By way of example, Figs. 92-95 show the graphs of the elementary functions $y = \sin^3 x$, $x \in \boldsymbol{R}$ (Fig. 92), $y = 2^{\sin x}$, $x \in \boldsymbol{R}$ (Fig. 93), $y = \sin(2^x)$, $x \in \boldsymbol{R}$ (Fig. 94), and $y = \log_2(\sin x)$ (Fig. 95).

With the help of elementary functions, we succeed in describing various physical processes and phenomena with sufficient precision. At the same time, elementary functions are rather simple since they are constructed on the basis of the twelve simplest elementary functions listed in Table 1, whose properties are known to us very well. All this represents the class of elementary functions as the main object for study in the course of mathematical analysis.

Exercises

1. For the functions $f(x) = x^2 + 3x - 1$, $g(x) = \log_2 x + 3$, and $h(x) = \sqrt{x} + \frac{1}{x^2+1}$ form $f(g(x))$, $f(h(x))$, $g(h(x))$, $g(f(x))$, $h(f(x))$, and $h(g(x))$.

2. Represent the given function in the form of a superposition of simpler functions:

(a) $y = \sqrt{x^2 + 2x + 3}$; (b) $y = \frac{1}{x^2+5x+2}$;

(c) $y = \sqrt{x - 2\sqrt{x}}$; (d) $y = \log_3(2x^2 - x + 3)$.

3. Prove the equality

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}, \quad x \in \boldsymbol{R}.$$

4. Sketch the graphs of the given functions:

(a) $y = \log_2 \cos x$; (b) $y = \log_2 |\cos x|$; (c) $y = \cos(\log_2 x)$.

5. Represent each of the given improper rational fraction as a sum of a polynomial and a proper rational fraction:

(a) $y = \frac{x+5}{x+1}$; (b) $y = \frac{x^2+3x+2}{x-1}$; (c) $y = \frac{x^5+x+1}{x^2+1}$.

6. Point out which of the given functions are irrational, rational, and transcendental functions and which are polynomials:

(a) $f(x) = 5 \sin \sqrt{x}$; (b) $f(x) = \sqrt{x^2 + 2x + 1}$;

(c) $f(x) = 3x^2 - 7x$; (d) $f(x) = \frac{2 \sin 3x}{x^2+2x-1}$;

(e) $f(x) = \frac{5}{x^3-5x^2+7}$; (f) $f(x) = (x^2 + 3) \cdot 2^x$.

5.4. Sequences

1. Numerical Sequences. An infinite numerical sequence is a numerical function f defined on the set of all natural numbers N. The values of such a function are denoted by a_n (or b_n, c_n, . . .). The number n is the ordinal number of the term a_n. A sequence is denoted by

$$\{a_n\}, \quad \text{or} \quad a_n, \quad n \in N, \quad \text{or} \quad a_1, a_2, \ldots, a_n, \ldots. \tag{1}$$

In other words, if every natural number n is associated with the number a_n, then we say that a sequence $\{a_n\}$ is defined. The number a_1 is called the first term of the sequence, a_2, the second, . . ., a_n, the nth (general) term of the sequence.

Let us recall the basic methods of defining an infinite sequence.

A sequence can be defined directly by giving the function f generating the sequence:

$$a_n = f(n), \quad n \in N. \tag{2}$$

Formula (2) enables us to compute the general term of the sequence a_n using the number n, for example:

$$a_n = 5^n, \quad n \in N, \tag{3}$$

$$a_n = \frac{1+(-1)^n}{2}, \quad n \in N. \tag{4}$$

Therefore formula (2) is called the formula of the general term of a sequence. Making use of this formula, we can compute any term of the sequence. For obviousness, the first several terms of a sequence are often written out in a line along with the formula for the general term. Thus, for sequence (4) we have:

$$a_1 = \frac{1+(-1)^2}{2} = \frac{1-1}{2} = 0, \quad a_2 = \frac{1+(-1)^2}{2} = 1,$$

$$a_3 = \frac{1+(-1)^3}{2} = 0,$$

and so on. Consequently, the given sequence has the form

$$0, 1, 0, 1, 0, 1, 0, \ldots.$$

Let us agree instead of the words: "Let us consider the sequence $\{a_n\}$ defined by the formula $a_n = f(n)$, $n \in N$" to formulate it in a shorter way: "Let us consider the sequence $a_n = f(n)$, $n \in N$" or still shorter: "Let $a_n = f(n)$".

Example 1. Let $a_n = \frac{(-1)^n}{n^3}$. Compute the first five terms of this sequence.

△ Using the formula of the general term of the sequence, we find

$$a_1 = \frac{(-1)^1}{1^3} = -1, \quad a_2 = \frac{(-1)^2}{2^3} = \frac{1}{8}, \quad a_3 = \frac{(-1)^2}{3^3} = -\frac{1}{27},$$

etc. This sequence has the form:

$$-1, \ \frac{1}{8}, \ -\frac{1}{27}, \ \frac{1}{64}, \ -\frac{1}{125}, \ \ldots . \ \blacktriangle$$

A sequence all terms of which attain equal values is called a constant sequence.

Example 2. Let $a_n = \cos(2\pi n)$, $n \in N$. Compute first several terms of the given sequence.

△ Using the formula of the general term of the sequence, we compute: $a_1 = \cos 2\pi = 1$, $a_2 = \cos 4\pi = 1$, $a_3 = \cos 6\pi = 1$, etc. The given sequence has the form 1, 1, 1, 1, . . ., that is, it is a constant sequence. ▲

Another widely spread method of defining a sequence is the recurrent method. This method consists in that a certain rule (usually a formula) is indicated which enables us to compute the general term of the sequence, using the previous terms, and several initial terms of the sequence are specified. The formula for calculating the general term of a sequence with the help of its previous terms is named the recurrence relation. The following formula may serve as an example of the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}. \tag{5}$$

Note that a given recurrence relation does not define a sequence completely. The thing is that the first terms of a sequence cannot be computed by means of the recurrence relation. For instance, formula (5) is meaningless for $n = 1$ and $n = 2$ since the terms a_0 and a_{-1} with the subscripts 0 and —1 do not exist, and, therefore, the values a_1 and a_2 should be defined additionally. Such values a_1 and a_2 are called initial for a given sequence. Further, beginning with a_3, the recurrence relation and the initial terms a_1 and a_2 will enable us to compute any term of the sequence under consideration.

Let, for instance, $a_1 = 1$, and $a_2 = 0$. Then

$$a_3 = 2a_2 - a_1 = -1, \quad a_4 = 2a_3 - a_2 = -2,$$

$$a_5 = 2a_4 - a_3 = -3,$$

etc. Thus, the sequence given by recurrent relation (5) and the initial terms $a_1 = 1$ and $a_2 = 0$ has the form:

$$1, \ 0, \ -1, \ -2, \ -3, \ -4, \ -5, \ \ldots .$$

Sometimes, a sequence can be defined "by words", that is, by describing its terms.

2. Bounded and Monotonic Sequences.

Definition 1. A sequence $\{a_n\}$ is said to be bounded if there exists a positive number M such that for any $n \in N$ the inequality $|a_n| \leqslant M$ is satisfied. Otherwise, it is said to be unbounded. For instance, the sequences $a_n = \frac{1}{n^4}$ and $a_n = (-1)^n$ are bounded since $0 < \frac{1}{n^4} \leqslant 1$, that is, $\left|\frac{1}{n^4}\right| \leqslant 1$, and $-1 \leqslant (-1)^n \leqslant 1$, that is, $|(-1)^n| \leqslant 1$.

Definition 2. A sequence $\{a_n\}$ is said to be increasing if for any n the inequality $a_{n+1} \geqslant a_n$ is satisfied.

If $a_{n+1} > a_n$, then it is said to be strictly increasing.

Definition 3. A sequence $\{a_n\}$ is said to be decreasing if for any n the inequality $a_{n+1} \leqslant a_n$ is valid.

If $a_{n+1} < a_n$, then it is said to be strictly decreasing.

All of the above sequences are called monotonic sequences. For example, the sequence $\left\{\frac{1}{n^3}\right\}$ is strictly decreasing since for any n we have $\frac{1}{(n+1)^3} < \frac{1}{n^3}$. The sequence $a_n = \frac{n-1}{n}$, $n \in N$, is strictly increasing since

$$a_{n+1} - a_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)} > 0,$$

and, consequently, $a_{n+1} > a_n$ for any n. The sequence 1, 1, 2, 2, 3, 3, ... is increasing, while 1, 1, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{3}$, ... is a decreasing sequence. Obviously, not any sequence is monotonic. For instance, the sequences

$$1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots \text{ and } 0, 1, 0, 1, 0, 1, \ldots$$

are not monotonic.

Exercises

1. Find the first six terms for each of the following sequences:

(a) $a_n = \frac{n^2}{n+1}$; (b) $a_n = \frac{2n^3-1}{n^3+1}$; (c) $a_n = \frac{(-1)^{n-1}}{n^3+3}$;

(d) $a_n = \frac{n}{3^n}$; (e) $a_n = \frac{(-1)^n}{n!}$.

2. Find the first six terms of the sequence $\{a_n\}$ if:

(a) $a_1 = 1$, $a_{n+1} = a_n + 2$; (b) $a_1 = 6$, $a_{n+1} = a_n - 2$;

(c) $a_1 = \frac{1}{5}$, $a_{n+1} = \frac{1}{a_n}$; (d) $a_1 = a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$.

3. Write the formula of the general term of the sequence whose first five terms coincide with the following:

(a) $3\cdot 3,\ 5\cdot 3^2,\ 7\cdot 3^3,\ 9\cdot 3^4,\ 11\cdot 3^5$;

(b) $\frac{1}{4},\ \frac{2}{4^2},\ \frac{3}{4^3},\ \frac{4}{4^4},\ \frac{5}{4^5}$;

(c) $\frac{1}{1\cdot 2},\ \frac{1}{2\cdot 3},\ \frac{1}{3\cdot 4},\ \frac{1}{4\cdot 5},\ \frac{1}{5\cdot 6}$;

(d) $\left(\frac{1}{3}\right)^2,\ \left(\frac{2}{5}\right)^2,\ \left(\frac{3}{7}\right)^2,\ \left(\frac{4}{9}\right)^2,\ \left(\frac{5}{11}\right)^2$;

(e) $1,\ \frac{1}{2\sqrt{2}},\ \frac{1}{3\sqrt{3}},\ \frac{1}{4\sqrt{4}},\ \frac{1}{5\sqrt{5}}$.

4. Determine which of the following sequences are monotonic, and which are not:

(a) $a_n=\frac{2n-3}{3n}$; (b) $a_n=\frac{n+5}{n}$; (c) $a_n=(-1)^{n-1}\cdot n-3$;

(d) $a_n=n^2-6n+5$; (e) $a_n=3n^2+5n+6$; (f) $a_n=\frac{2-n^2}{n^2}$;

(g) $a_n=\frac{4n^2+2}{n^2}$; (h) $a_n=\frac{(-1)^{n-1}}{n}$.

5. Which of the given sequences are bounded, and which are not:

(a) $a_n=\frac{1}{2n}$; (b) $a_n=\frac{2n}{3n-1}$; (c) $a_n=(-1)^n\cdot n$;

(d) $a_n=(-1)^n\cdot n^3$; (e) $a_n=\frac{(-1)^{n-1}}{2n}$; (f) $a_n=\frac{2n}{3n+6}$;

(g) $a_n=4^n$; (h) $a_n=2^{-n}$; (i) $a_n=n^n$?

6. A sequence $\{a_n\}$ is given recurrently:

(a) $a_1=0,\ a_{n+1}=\frac{a_n+1}{4}$; (b) $a_1=0,\ a_2=1,\ a_{n+1}=\frac{a_n+a_{n-1}}{2}$.

Prove that these sequences are bounded.

5.5. The Limit of a Sequence

1. The Limit of a Numerical Sequence. Convergent and Divergent Sequences. Consider the sequence

$$b_n=\frac{8n-1}{n},\quad n\in N. \tag{1}$$

It is easy to see that $b_n=8-\frac{1}{n}$ and, consequently, b_n approaches arbitrarily close to 8 as the number n of its terms increases. Let us set the following problem: to formulate this assertion in a precise mathematical way. To this end, we shall first answer the following

question: what value must be assigned to n in order to obtain the absolute value of the difference $b_n - 8$ equal to less than 0.001? Since

$$|b_n - 8| = \frac{1}{n},$$

the inequality $| b_n - 8 | < 0.001$ is satisfied for any $n > N = 1000$. For an arbitrary positive number ε the inequality

$$| b_n - 8 | \leqslant \varepsilon \tag{2}$$

is equivalent to the inequality $n > \frac{1}{\varepsilon}$. Since we are interested in natural values of n, we find that inequality (2) is satisfied for any $n > N = \left[\frac{1}{\varepsilon}\right]$, where $\left[\frac{1}{\varepsilon}\right]$ is the integral part of the number $\frac{1}{\varepsilon}$. In this case, we say that the limit of sequence (1) is equal to 8 and write

$$\lim_{n \to \infty} \frac{8n-1}{n} = 8.$$

Let us now formulate the corresponding definition.

Definition 1. Let a numerical sequence $\{a_n\}$ be given. The number a is called the limit of this sequence if for any given number $\varepsilon > 0$ there is a natural number N such that for any $n > N$ the following inequality is satisfied:

$$| a_n - a | < \varepsilon.$$

In this case, we write:

$$\lim_{n \to \infty} a_n = a \quad \text{or} \quad a_n \to a \quad \text{as} \quad n \to \infty$$

which is read: "The sequence $\{a_n\}$ has the number a as its limit" or "The sequence $\{a_n\}$ converges to the number a".

The above example shows that the choice of the natural number N depends on the given positive number ε. To note this dependence, we write: $N = N(\varepsilon)$ or $N = N_\varepsilon$.

Definition 2. A sequence which has a limit is convergent; otherwise it is divergent.

Let us consider a couple of examples on computing the limit of a sequence using the definition of the limit.

Example 1. Prove that

$$\lim_{n \to \infty} \frac{4n+1}{3n-1} = \frac{4}{3}. \tag{3}$$

△ By definition, the number $\frac{4}{3}$ will be the limit of this sequence if for each number $\varepsilon > 0$ there is a natural number $N = N(\varepsilon)$ such

that for all $n > N$ the following inequality is satisfied:

$$\left| \frac{4n+1}{3n-1} - \frac{4}{3} \right| = \frac{7}{3(3n-1)} < \varepsilon.$$

It is valid for all $n > \frac{7+3\varepsilon}{9\varepsilon}$, that is, for all $n > N(\varepsilon)$, where $N(\varepsilon) = \left[\frac{7+3\varepsilon}{9\varepsilon} \right]$. This just proves equality (3). ▲

Example 2. Prove that

$$\lim_{n \to \infty} \frac{1}{10^n} = 0. \tag{4}$$

△ The inequality $\left| \frac{1}{10^n} - 0 \right| = \frac{1}{10^n} < \varepsilon$ is equivalent to the inequality $n > \log\left(\frac{1}{\varepsilon}\right)$. Thus, for any positive ε the inequality $\left| \frac{1}{10^n} - 0 \right| < \varepsilon$ is satisfied for all $n > N(\varepsilon) = \left[\log\left(\frac{1}{\varepsilon}\right) \right]$. This just proves equality (4). ▲

Theorem 5.5.1. *If a sequence has a limit, then it is bounded.*

□ Let a sequence $\{a_n\}$ has the number a as its limit. Let us prove that it is bounded. We take some number $\varepsilon > 0$. Then there is a number N such that only first N terms of the sequence $a_1, a_2, \ldots, a_N$ may be found outside the interval $(a - \varepsilon, a + \varepsilon)$. Among the numbers $a_1, a_2, \ldots, a_N, a - \varepsilon, a + \varepsilon$ we shall find the least and the greatest numbers and denote them by m and M, respectively. Then $m \leqslant a_n \leqslant M$ for all n, and this just means that the sequence $\{a_n\}$ is bounded. ■

Theorem 5.5.2. *Any convergent sequence has only one limit.*

□ We shall use here the method of indirect proof. Suppose that a convergent sequence $\{a_n\}$ has two different limits a and b. Let, for definiteness, $a < b$. Setting $\varepsilon = \frac{b-a}{3} > 0$, we get

$$a + \varepsilon < b - \varepsilon. \tag{5}$$

Since $\lim\limits_{n \to \infty} a_n = a$, then, according to the definition of the limit, for the chosen $\varepsilon = \frac{b-a}{3}$ there is a number N_1 such that for all $n > N_1$ we shall have $|a_n - a| < \varepsilon$ and, in particular, $a_n < a + \varepsilon$. On the other hand, since $\lim\limits_{n \to \infty} a_n = b$, there is a number N_2 such that for all $n > N_2$ we shall have $|a_n - b| < \varepsilon$ and, in particular, $b - \varepsilon < a_n$. Putting $N = \max\{N_1, N_2\}$ (that is, choosing the greater of the two numbers N_1 and N_2), we get $b - \varepsilon < a_n < a + \varepsilon$ for $n > N$, but this contradicts inequality (5). Consequently, a sequence cannot have two different limits. ■

2. Infinitely Small Sequences. The computation of limits and proofs of the theorems on limits are simplified by introducing the notion of infinitely small sequence.

Definition. A sequence is called infinitely small if its limit is equal to zero. For instance, the sequence $a_n = \frac{1}{n}$ is infinitely small since its limit is equal to zero. The sequence $a_n = \frac{1}{10^n}$ is also infinitely small since $\lim\limits_{n\to\infty} \frac{1}{10^n} = 0$ (see Example 2 from the preceding subsection).

Theorem 5.5.3. *The sum of two infinitely small sequences is an infinitely small sequence.*

□ Let $\{a_n\}$ and $\{b_n\}$ be two infinitely small sequences. Then for any $\varepsilon > 0$ there is a number N_1 such that

$$|a_n| < \frac{\varepsilon}{2} \tag{1}$$

for all $n > N_1$, and there is a number N_2 such that

$$|b_n| < \frac{\varepsilon}{2} \tag{2}$$

for all $n > N_2$. Taking $N = \max\{N_1, N_2\}$, we obtain that for any $n > N$ inequalities (1) and (2) are simultaneously valid. Consequently, for any $n > N$ we get

$$|a_n + b_n| \quad |a_n| + |b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was taken arbitrarily, we have proved thereby that $\lim\limits_{n\to\infty} (a_n + b_n) = 0$, that is, the sequence $\{a_n + b_n\}$ is infinitely small. ■

It is proved in a similar way that the sum of any finite number of infinitely small sequences is an infinitely small sequence.

Theorem 5.5.4. *The product of an infinitely small sequence by a bounded sequence is an infinitely small sequence.*

□ Let $\{b_n\}$ be a bounded sequence, that is,

$$|\, b_n \,| \leqslant M \tag{3}$$

for all n. Let $\{a_n\}$ be an infinitely small sequence. Then for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that

$$|a_n| < \frac{\varepsilon}{M+1} \tag{4}$$

for all $n > N$. From inequalities (3) and (4) it follows that

$$|a_n b_n| = |a_n| \cdot |b_n| \leqslant \frac{\varepsilon}{M+1} \cdot M < \varepsilon$$

for any $n > N$. And this just means that

$$\lim_{n\to\infty} a_n b_n = 0,$$

that is, the sequence $\{a_n b_n\}$ is infinitely small. ■

Since an infinitely small sequence is bounded, it follows from Theorem 5.5.4 that the product of two infinitely small sequences is an infinitely small sequence.

Theorem 5.5.5. *For the number a to be the limit of a sequence $\{a_n\}$, it is necessary and sufficient that a_n be represented as $a_n = a + \alpha_n$, where $\{\alpha_n\}$ is an infinitely small sequence, that is, $\alpha_n \to 0$ as $n \to \infty$.*

□ By hypothesis $\lim\limits_{n\to\infty} a_n = a$. Then for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$ such that

$$|a_n - a| < \varepsilon \tag{5}$$

for all $n > N$. Let $\alpha_n = a_n - a$, then

$$|\alpha_n| < \varepsilon \tag{6}$$

for all $n > N$, whence it follows that $\lim\limits_{n\to\infty} \alpha_n = 0$. Thus, if the number a is the limit of the sequence $\{a_n\}$, then $a_n = a + \alpha_n$, where $\{\alpha_n\}$ is an infinitely small sequence. The converse is proved in a similar way since (5) follows from (6). ■

Example. Evaluate the limit $\lim\limits_{n\to\infty} \dfrac{n+\sin n}{n}$.

△ We denote $a_n = \dfrac{n+\sin n}{n}$ and $b_n = \dfrac{\sin n}{n}$. Since $a_n = 1 + b_n$, by Theorem 5.5.5, the number 1 will be the limit of the sequence $\{a_n\}$ if the sequence $\{b_n\}$ is infinitely small. According to Theorem 5.5.4, the product of the bounded sequence $\{\sin n\}$ by the infinitely small sequence $\left\{\dfrac{1}{n}\right\}$, that is, by the sequence $\{b_n\}$, is an infinitely small sequence. Thus, the number 1 is the limit of the sequence $\{a_n\}$, that is,

$$\lim_{n\to\infty} \frac{n+\sin n}{n} = 1. \ ▲$$

3. Theorems on the Limits of Sequences Involving Arithmetical Operations. The following theorems are frequently used in computing the limits of sequences.

Theorem 5.5.6 (*on the limit of a sum*). *If two sequences $\{a_n\}$ and $\{b_n\}$ converge, then their sum $\{a_n + b_n\}$ also converges, and the limit of the sum is equal to the sum of the limits:*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

□ Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$. Then by Theorem 5.5.5, we have:

$$a_n = a + \alpha_n, \quad b_n = b + \beta_n,$$

where $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Consequently,

$$a_n + b_n = (a + b) + (\alpha_n + \beta_n),$$

where $(\alpha_n + \beta_n) \to 0$ as $n \to \infty$, since the sum of two infinitely small sequences is an infinitely small sequence. Then, by Theorem 5.5.5, we have:

$$\lim_{n\to\infty} (a_n + b_n) = a + b = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n. \blacksquare$$

Theorem 5.5.7 (*on the limit of a product*). *If two sequences* $\{a_n\}$ *and* $\{b_n\}$ *converge, then their product* $\{a_n b_n\}$ *also converges, and the limit of the product is equal to the product of the limits*:

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n).$$

□ Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$. Then, by Theorem 5.5.5,

$$a_n = a + \alpha_n, \quad b_n = b + \beta_n,$$

where $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to \infty$. Multiplying the last two equalities, we get

$$a_n b_n = ab + (b\alpha_n + a\beta_n + \alpha_n\beta_n).$$

From Theorems 5.5.3 and 5.5.4 it follows that $(b\alpha_n + a\beta_n + \alpha_n\beta_n) \to 0$ as $n \to \infty$. Then, by Theorem 5.5.5,

$$\lim_{n\to\infty} (a_n b_n) = ab = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n). \blacksquare$$

Corollary 1. A constant factor may be taken outside the sign of limit:

$$\lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n.$$

□ By Theorem 5.5.7, we have $\lim\limits_{n\to\infty} (ca_n) = (\lim\limits_{n\to\infty} c)(\lim\limits_{n\to\infty} a_n) = c \lim\limits_{n\to\infty} a_n$, since $\lim\limits_{n\to\infty} c = c$. ■

Corollary 2. If the sequences $\{a_n\}$ and $\{b_n\}$ converge, then their difference $\{a_n - b_n\}$ also converges, and the limit of the difference is equal to the difference of the limits:

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

□ By Theorem 5.5.3 and Corollary 1, we have

$$\lim_{n\to\infty}(a_n-b_n)=\lim_{n\to\infty}(a_n+(-1)b_n)=\lim_{n\to\infty}a_n+\lim_{n\to\infty}((-1)b_n)=$$

$$\lim_{n\to\infty}a_n-\lim_{n\to\infty}b_n.\ \blacksquare$$

Given below without proof is the theorem on the limit of a quotient.

Theorem 5.5.8 (*on the limit of a quotient*). *If two sequences* $\{a_n\}$ *and* $\{b_n\}$ *converge, and* $b_n\neq 0$ *and* $\lim\limits_{n\to\infty}b_n\neq 0$, *then their quotient* $\left\{\frac{a_n}{b_n}\right\}$ *converges, and the limit of the quotient is equal to the quotient of the limits*:

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}.$$

Consider a couple of examples illustrating how limits are computed with the help of the above theorems.

Example 1. Find the limit

$$\lim_{n\to\infty}\frac{7-5n}{3-4n}.$$

△ Since both the numerator and denominator represent divergent sequences (as they are unbounded), it is impossible here to directly apply the theorem on the limit of a quotient. However, we may first divide both the numerator and the denominator by n (the fraction will remain unchanged) and then apply the theorem on the limit of a quotient and a difference:

$$\lim_{n\to\infty}\frac{7-5n}{2-4n}=\lim_{n\to\infty}\frac{\frac{7}{n}-5}{\frac{3}{n}-4}=\frac{\lim\limits_{n\to\infty}\left(\frac{7}{n}-5\right)}{\lim\limits_{n\to\infty}\left(\frac{3}{n}-4\right)}=\frac{5}{4}.\ \blacktriangle$$

Example 2. Evaluate the limit

$$\lim_{n\to\infty}\frac{5(n^2+2n+1)}{1-n-n^2}.$$

△ We have

$$\lim_{n\to\infty}\frac{5(n^2+2n+1)}{1-n-n^2}=5\lim_{n\to\infty}\frac{n^2+2n+1}{1-n-n^2}$$

$$=5\lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{1}{n^2}}{\frac{1}{n^2}-\frac{1}{n}-1}=5\cdot\frac{1}{(-1)}=-5.\ \blacktriangle$$

4. Theorems on Limits in Inequalities. When proving certain theorems which will be encountered later and solving a number of problems, it is convenient to use the following theorem.

Theorem 5.5.9 (*on three sequences*). *If the terms of three sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are related by the inequalities* $a_n \leqslant c_n \leqslant b_n$, $n \in N$, *and the limits of the sequences* $\{a_n\}$ *and* $\{b_n\}$ *exist and are equal to each other, then the limit of the sequence* $\{c_n\}$ *exists and is equal to the common limit of* $\{a_n\}$ *and* $\{b_n\}$.

□ Let $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = a$. According to the definition of the limit of a sequence, for any $\varepsilon > 0$ there is a number N_1 such that

$$a - \varepsilon < a_n < a + \varepsilon$$

for all $n > N_1$, and there is a number N_2 such that

$$a - \varepsilon < b_n < a + \varepsilon$$

for all $n > N_2$. Then for $n > N = \max\{N_1, N_2\}$ we have:

$$a - \varepsilon < a_n \leqslant c_n \leqslant b_n < a + \varepsilon,$$

whence

$$|c_n - a| < \varepsilon \tag{1}$$

for all $n > N$. Thus, for any $\varepsilon > 0$ there exists a number N such that for all $n > N$ inequality (1) is satisfied. And this just means that $\lim\limits_{n\to\infty} c_n = a$. ■

Example. Find the limit

$$\lim_{n\to\infty} \frac{5^n}{n^n}.$$

△ Since for $n \geqslant 15$ the inequality $\frac{5}{n} \leqslant \frac{1}{3}$ is true, we have

$$0 < \left(\frac{5}{n}\right)^n \leqslant \left(\frac{1}{3}\right)^n$$

for $n \geqslant 15$. Here both on the left and on the right are the terms of the sequences having zero as limit (see Example 2 from subsection 1). Hence, by the theorem on three sequences,

$$\lim_{n\to\infty} \left(\frac{5}{n}\right)^n = 0. \ ▲$$

Theorem 5.5.10. *Let two sequences* $\{a_n\}$ *and* $\{b_n\}$ *converge to a and b, respectively. Then, if* $a < b$, *there is a number N such that* $a_n < b_n$ *for all* $n > N$.

□ We set $\varepsilon = \frac{b-a}{a}$. Then there is a number N_1 such that for all $n > N_1$ the inequality $|a_n - a| < \varepsilon$ is valid, and, in particular,

$$a_n < a + \varepsilon = \frac{a+b}{2}. \tag{2}$$

Analogously, there exists a number N_2 such that $|b_n - b| < \varepsilon$ for all $n > N_2$, and, in particular,

$$b_n > b - \varepsilon = \frac{a+b}{2}. \tag{3}$$

It follows from (2) and (3) that $a_n < b_n$ for all $n > N = \max\{N_1, N_2\}$. ■

Theorem 5.5.11. *Let the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively. If there is a number N such that $a_n \leqslant b_n$ for all $n > N$, then $a \leqslant b$.*

□ We shall use here the method of indirect proof. Let us assume that $a > b$. Then, by virtue of Theorem 5.5.10, there exists a number N_1 such that $a_n > b_n$ for all $n > N_1$, but this contradicts the hypothesis. Consequently, our assumption is false. ■

Corollary. If the sequence $\{a_n\}$ converges, and $a_n \leqslant c$ for all $n \in N$, then $\lim\limits_{n\to\infty} a_n \leqslant c$.

Theorem 5.5.11 states that it is possible to pass to the limits in inequalities, provided the limits of both sides of an inequality exist. Namely, the conditional inequality $a_n \leqslant b_n$ changes into the conditional inequality $a \leqslant b$. Note that in the limit the strict inequality may change into the equality $a = b$. For instance, $\frac{1}{n} > 0$ for any $n \in N$, but $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

5. Infinitely Large Sequences.

Definition. A sequence $\{a_n\}$ is called infinitely large if for any $A > 0$ there is a number N_A such that for all $n > N_A$ the inequality $|a_n| > A$ holds true. In this case, we write:

$$\lim_{n\to\infty} a_n = \infty.$$

For instance, $\lim\limits_{n\to\infty} (2n) = \infty$, that is, the sequence $\{2n\}$ is infinitely large.

Theorem 5.5.12. *If the sequence $\{a_n\}$, where $a_n \neq 0$, is infinitely large, then the sequence $\left\{\frac{1}{a_n}\right\}$ is infinitely small, and vice versa: if the sequence $\{a_n\}$ is infinitely small, then the sequence $\left\{\frac{1}{a_n}\right\}$ is infinitely large.*

□ By hypothesis, $\{a_n\}$ is an infinitely large sequence. By definition, for any $A > 0$ there is a number N such that

$$|a_n| > A \tag{1}$$

for all $n > N$. Setting $\varepsilon = \frac{1}{A}$, from inequality (1) we get

$$\left|\frac{1}{a_n}\right| = \frac{1}{|a_n|} < \frac{1}{A} = \varepsilon$$

for all $n > N$. Since ε may be an arbitrary positive number here, this just means that $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$, that is, $\left\{\frac{1}{a_n}\right\}$ is an infinitely small sequence.

Let us now prove the converse. Let $\{a_n\}$ be an infinitely small sequence. Then, by definition, for any $\varepsilon > 0$ there is a number N such that

$$|a_n| < \varepsilon \tag{2}$$

for all $n > N$. Setting $A = \frac{1}{\varepsilon} > 0$, from relation (2) we get

$$\frac{1}{|a_n|} > \frac{1}{\varepsilon} = A$$

for all $n > N$. Since A is an arbitrary positive number, this just means that the sequence $\left\{\frac{1}{a_n}\right\}$ is infinitely large. ■

Example. Show that $\lim\limits_{n\to\infty} q^n = 0$ if $|q| < 1$, and $\lim\limits_{n\to\infty} q^n = \infty$ if $|q| > 1$.

△ Using the method of mathematical induction, let us first prove the inequality

$$(1 + \alpha)^n \geqslant 1 + n\alpha, \quad \alpha > -1, \quad n \in N. \tag{3}$$

This inequality is called Bernoulli's inequality.

For $n = 1$ conditional inequality (3) occurs: $1 + \alpha \geqslant 1 + \alpha$. Let us now assume that (3) is true for some $n = k$ and prove that in such a case it will also be true for $n = k + 1$. Thus, let

$$(1 + \alpha)^k \geqslant 1 + k\alpha.$$

Multiplying both sides of the inequality by $(1 + \alpha)$ $(1 + \alpha > 0$, since $\alpha > -1)$, we get:

$$(1 + \alpha)^{k+1} \geqslant (1 + k\alpha)(1 + \alpha) = 1 + (k + 1)\alpha + k\alpha^2 \geqslant 1 + (k + 1)\alpha,$$

hence inequality (3) is also true for $n = k + 1$. Thereby, we have proved the validity of (3) for any n by the method of mathematical induction.

Consider the sequence $\{q^n\}$ for $|q| > 1$. In this case, $|q| = 1 + h$, where $h > 0$. According to Bernoulli's inequality,

$$|q|^n = (1 + h)^n \geqslant 1 + hn. \tag{4}$$

It is obvious that

$$\lim_{n\to\infty}(1+hn)=\infty,$$

that is, the sequence $\{1+hn\}$ is infinitely large. Inequality (4) implies that the sequence $\{q^n\}$ is also infinitely large, that is,

$$\lim_{n\to\infty} q^n=\infty$$

if $|q|>1$.

Let now $|q|<1$. If $q=0$, then $q^n=0$ for any $n\in N$, and

$$\lim_{n\to\infty} q^n=0.$$

But if $q\neq 0$, then $\left|\frac{1}{q}\right|>1$. According to what has been proved, the sequence $\left(\frac{1}{q}\right)^n$, $n\in N$, is infinitely large. Then, by virtue of the theorem on the relation between an infinitely large and an infinitely small sequences, the sequence q^n, $n\in N$, is infinitely small, that is, $\lim\limits_{n\to\infty} q^n=0$ if $|q|<1$. ▲

6. On Existence of the Limit of a Monotonic Sequence. The Number *e*. Consider the sequence

$$b_n=\left(1+\frac{1}{n}\right)^{n+1} \tag{1}$$

To prove the convergence of sequence (1), let us make use of the following theorem:

Weierstrass' theorem. *Any monotonic bounded sequence has a limit.*

Let us prove that sequence (1) decreases and is bounded. Since for $n\geqslant 2$

$$\frac{b_{n-1}}{b_n}=\frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}}=\frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^{n+1}}=\frac{n^{2n+1}}{(n+1)^{n+1}(n-1)^n}$$

$$=\frac{(n^2)^{n+1}}{(n^2-1)^{n+1}}\cdot\frac{n-1}{n}=\left(\frac{n^2}{n^2-1}\right)^{n+1}\cdot\frac{n-1}{n}$$

$$=\left(1+\frac{1}{n^2-1}\right)^{n+1}\cdot\frac{n-1}{n}$$

and, according to Bernoulli's inequality,

$$\left(1+\frac{1}{n^2-1}\right)^{n+1}\geqslant 1+(n+1)\frac{1}{n^2-1}=1+\frac{1}{n-1}=\frac{n}{n-1},$$

we have

$$\frac{b_{n-1}}{b_n}\geqslant\frac{n}{n-1}\cdot\frac{n-1}{n}=1.$$

Thus, the sequence $\{b_n\}$ decreases. And since

$$0 < b_n \leqslant b_1 = 4$$

for any $n \in N$, the sequence is bounded. Then, by Weierstrass' theorem, it has a limit.

This limit is customarily denoted by the letter e, so that, by definition,

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1}.$$

To determine the number e, we often use the sequence

$$a_n = \left(1+\frac{1}{n}\right)^n, \quad n \in N.$$

Since

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1,$$

we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{b_n}{\left(1+\frac{1}{n}\right)} = e$. Thus,

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

The number e plays an important part in mathematics. Here, we give only the definition of the number e but do not indicate the method of computing this number within any degree of accuracy. This will be done in the next chapter. For the time being we note that the number e is an irrational number and, to within 10^{-9}, it is equal to 2.718281828.

Exercises

1. Prove that

(a) $\lim\limits_{n\to\infty} \dfrac{4n-8}{2n} = 2$; (b) $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{5n} = 0$;

(c) $\lim\limits_{n\to\infty} \dfrac{2a-5}{4n+17} = \dfrac{1}{2}$; (d) $\lim\limits_{n\to\infty} \dfrac{1}{3n^2+1} = 0$.

2. Determine which of the given sequences $\{a_n\}$ are convergent, and which are divergent:

(a) $a_n = \dfrac{3n+6}{5n}$; (b) $a_n = \dfrac{(-5)^n+4}{4}$; (c) $a_n = 4^n - 1$;

(d) $a_n = n^2 - 1$; (e) $a_n = \dfrac{(-1)^n}{n^6}$.

3. Evaluate the given limits:

(a) $\lim\limits_{n\to\infty} \frac{3n-3}{7-8n}$; (b) $\lim\limits_{n\to\infty} \frac{1}{4^n}$; (c) $\lim\limits_{n\to\infty} \frac{5-n}{3n+1}$;

(d) $\lim\limits_{n\to\infty} \frac{6n^2+2}{1-7n^2}$; (e) $\lim\limits_{n\to\infty} \frac{1-2n-3n^3}{(3n+1)^3}$;

(f) $\lim\limits_{n\to\infty} \left(\frac{1}{2n}+\frac{2n}{3n+1}\right)$; (g) $\lim\limits_{n\to\infty} \frac{2n^2+5}{4n^2+6n-7}$.

4. Which of the following statements are true:
(a) if a sequence is unbounded, then it has no limit;
(b) if a sequence has a limit, then it is bounded;
(c) if a sequence is not monotonic, then it has no limit;
(d) if a sequence has a limit, then it is monotonic?

5. Find out whether the given sequence has a limit and evaluate it if any:

(a) $a_n = -\frac{1}{5n}$; (b) $a_n = \frac{7}{7n-5}$; (c) $a_n = \frac{3n+1}{4n^2+3}$;

(d) $a_n = \frac{1}{6n}$; (e) $a_n = \frac{1}{3^n}$; (f) $a_n = \frac{1}{n-(-1)^n}$;

(g) $a_n = n-(-1)^{n+1}$; (h) $a_n = \frac{1}{\sqrt{6n-8}}$.

5.6. The Limit of a Function

1. The Limit of a Function at a Point. Let us give a definition of the limit of a function at a point.

Definition. Let a function $f(x)$ be defined in some neighbourhood of the point a, except, possibly, for the point a itself. The number B is called the limit of the function $f(x)$ at the point a (or as x tends to a) if for any sequence of permissible values of the argument x_n, $n \in N$, $x_n \neq a$, converging to a (that is $\lim\limits_{n\to\infty} x_n = a$), the sequence of the corresponding values of the function $f(x_n)$, $n \in N$ converges to the number B. In this case we write:

$$\lim_{x\to a} f(x) = B \text{ or } f(x) \to B \text{ as } x \to a.$$

We can put it in a shorter way: $B = \lim\limits_{x=a} f(x)$ if $\lim\limits_{n\to\infty} f(x_n) = B$ for any sequence of permissible values of the argument $x_n \neq a$, $n \in N$, converging to a.

But if for some sequence of values of the argument $x_n \neq a$, $n \in N$, converging to a the sequence of the corresponding values of the function $f(x_n)$, $n \in N$, has no limit, then the function $f(x)$ has no limit at the point a. Analogously, the function $f(x)$ has no limit at point a

if for two different sequences of values of the argument which converge to a the sequences of the corresponding values of the function have different limits.

It follows from the definition of the limit that if $f(x) \to B$ as $x \to a$, then $f(x) - B \to 0$ as $x \to a$. Denoting $f(x) - B$ by $\alpha(x)$, we get $f(x) = B + \alpha(x)$, where $\alpha(x) \to 0$ as $x \to a$. It is obvious that the number B is the limit of the function $f(x)$ as $x \to a$ if and only if $f(x)$ is representable in the form

$$f(x) = B + \alpha(x),$$

where $\alpha(x) \to 0$ as $x \to a$.

Note that the point a at which the limit of the function $f(x)$ has been considered may belong to the domain of definition of the function $f(x)$ and it may not, but it must always be the limiting point of the domain of definition of the function. When finding the limit of a function at a point, the value of the function at this point is not considered.

Using the definition of the limit, let us find the limits of some functions.

Example 1. Prove that the limit of a constant function is equal to the same constant.

△ Let $f(x) = c$ for all x from some interval containing the point a. Then for any sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_n = a$, we have $f(x_n) = c$ and $\lim\limits_{n\to\infty} f(x_n) = c$. Consequently, $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} c = c$. ▲

Example 2. Prove that for all $f(x) = x$

$$\lim_{x\to a} f(x) = \lim_{x\to a} x = a.$$

△ For any sequence $\{x_n\}$ such that $x_n \to a$ as $n \to \infty$, we have $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n = a$. Consequently, according to the definition of the limit of a function at a point $\lim\limits_{x\to a} x = a$. ▲

Example 3. Prove that the function $f(x) = \sin\dfrac{1}{x}$ has no limit as $x \to 0$.

△ Consider three sequences of permissible values of the argument:

$$x_n = \frac{1}{\pi n}, \quad x'_n = \frac{1}{\frac{\pi}{2} + 2\pi n}, \quad x''_n = \frac{1}{\frac{\pi}{2} + \pi n}, \quad n \in N.$$

It is obvious that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = \lim_{n\to\infty} x''_n = 0.$$

For the first sequence we get

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(\pi n) = \lim_{n\to\infty} 0 = 0.$$

For the second sequence we find

$$\lim_{n\to\infty} f(x_n') = \lim_{n\to\infty} \sin\left(\frac{\pi}{2} + 2\pi n\right) = \lim_{n\to\infty} 1 = 1.$$

For the third sequence we have $f(x_n'') = \sin\left(\frac{\pi}{2} + \pi n\right) = (-1)^n$, that is, the sequence of values of the function has the form:

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \ldots$$

and has no limit as $n \to \infty$. Hence it follows that, according to the definition of the limit of a function at a point, the function $f(x) = \sin\frac{1}{x}$ has no limit as $x \to 0$. ▲

Theorem 5.6.1. *A function cannot have two different limits at one point.*

□ We shall use here the method of indirect proof. Let us assume that at the point a the function $f(x)$ has two different limits A and B. By the definition of the limit, for any sequence of values of the argument $\{x_n\}$, $n \in N$, such that $x_n \neq a$ and $\lim\limits_{n\to\infty} x_n = a$, we have:

$$\lim_{n\to\infty} f(x_n) = A, \quad \lim_{n\to\infty} f(x_n) = B.$$

Hence, by virtue of the uniqueness of the limit of a sequence, we get the equality $A = B$ which contradicts our assumption. Consequently, a function cannot have two different limits at a point. ■

2. Theorems on the Limits of Functions Involving Arithmetical Operations. The basic theorems on the limits of functions (on the limit of a sum, of a difference, of a product, and of a quotient) which facilitate the computation of limits are similar to the corresponding theorems on the limits of sequences.

Theorem 5.6.2. *The limit of a sum (difference) of two functions is equal to the sum (difference) of their limits if the latter exist:*

$$\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$

Theorem 5.6.3. *The limit of a product of functions is equal to the product of their limits if the latter exist:*

$$\lim_{x\to a} (f(x)\, g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x).$$

Corollary. A constant factor can be taken outside the sign of limit, that is,

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x),$$

provided that $\lim\limits_{x\to a} f(x)$ exists.

Theorem 5.6.4. *The limit of a ratio of two functions is equal to the ratio of their limits if the latter exist, and the limit of the denominator is nonzero:*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)},$$

if $\lim\limits_{x\to a} g(x) \neq 0$.

□ Let $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to a} g(x) = B \neq 0$. According to the definition of the limit of a function at a point for any sequence $\{x_n\}$ of values of the argument such that $x_n \neq a$ and $\lim\limits_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = A \quad \text{and} \quad \lim_{n\to\infty} g(x_n) = B \neq 0.$$

Using these equalities and the theorem on the limit of a quotient for convergent sequences, we get

$$\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)} = \frac{\lim\limits_{n\to\infty} f(x_n)}{\lim\limits_{n\to\infty} g(x_n)} = \frac{A}{B}.$$

Hence it follows that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{A}{B}, \text{ i.e. } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)},$$

which was required to be proved. ■

Theorems 5.6.2 and 5.6.3 are proved in a similar way.

Example 1. Evaluate the limit $\lim\limits_{x\to 1}(6x^3 - 3x^2 + 5)$.

△ Applying the theorems on the limits of a sum, difference, and product, we get $\lim\limits_{x\to 1}(6x^3 - 3x^2 + 5) = \lim\limits_{x\to 1}(6x^3) - \lim\limits_{x\to 1}(3x^2) + \lim\limits_{x\to 1} 5 = 6\lim\limits_{x\to 1} x^3 - 3\lim\limits_{x\to 1} x^2 + 5 = 6(\lim\limits_{x\to 1} x)(\lim\limits_{x\to 1} x)(\lim\limits_{x\to 1} x) - 3(\lim\limits_{x\to 1} x)(\lim\limits_{x\to 1} x) + 5 = 6\cdot 1\cdot 1\cdot 1 - 3\cdot 1\cdot 1 + 5 = 8$. ▲

Example 2. Evaluate $\lim\limits_{x\to 3}\dfrac{x^2 - 5x + 6}{x - 3}$.

△ Here the limit of the denominator is equal to zero, therefore it is impossible to make use of the theorem on the limit of a quotient. Let us factor the numerator into linear factors:

$$x^2 - 5x + 6 = (x - 3)(x - 2).$$

Since, when finding the limit at the point $x = 3$, we consider only $x \neq 3$, it is possible to reduce the fraction by $(x - 3)$, and therefore

$$\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \lim_{x\to 3} \frac{(x-2)(x-3)}{x-3} = \lim_{x\to 3} (x-2) = 3-2 = 1. \ ▲$$

3. Theorems on the Limits of Functions in Inequalities.

Theorem 5.6.5 (*on the limit of an intermediate function*). *If*

$$\lim_{x\to a} \varphi(x) = B, \quad \lim_{x\to a} \psi(x) = B$$

and in some neighbourhood of the point a except perhaps for the point a itself the inequalities

$$\varphi(x) \leqslant f(x) \leqslant \psi(x),$$

are satisfied, then

$$\lim_{x\to a} f(x) = B.$$

Theorem 5.6.6. *If a function $f(x)$ has a limit at the point $x = a$, and $f(x) \leqslant B$ for all x from some neighbourhood of the point a, then*

$$\lim_{x\to a} f(x) \leqslant B.$$

These theorems follows from the corresponding theorems for sequences.

Example. Prove that $\lim\limits_{x\to 2} \sqrt{x} = \sqrt{2}$.

△ For any $x > 0$ we have $0 \leqslant |\sqrt{x} - \sqrt{2}| = \dfrac{|x-2|}{\sqrt{x}+\sqrt{2}} \leqslant |x-2|$. Since $\lim\limits_{x\to 2} 0 = \lim\limits_{x\to 2} |x-2| = 0$, by the theorem on the limit of an intermediate function, we have:

$$\lim_{x\to 2} |\sqrt{x} - \sqrt{2}| = 0 \text{ or } \lim_{x\to 2} (\sqrt{x} - \sqrt{2}) = 0,$$

whence $\lim\limits_{x\to 2} \sqrt{x} = \sqrt{2}$. ▲

4. One-sided (Unilateral) Limits. In the definition of the limit of a function at a point (see subsection 1 of the present section) the argument x takes on the values x_n from the neighbourhood of the point a, except for $x = a$, both on the left and on the right of a.

If, when finding a limit, we consider the values of x only on the left of a, then such a limit is called the left-hand limit (or limit on the left) and is denoted by

$$\lim_{x \to a-0} f(x) = \lim_{\substack{x \to a \\ x<a}} f(x) = f(a-0),$$

and if the values of x are considered only on the right of the point a, then such a limit is termed the right-hand limit (or limit on the right) and is denoted as

$$\lim_{x \to a+0} f(x) = \lim_{\substack{x \to a \\ x>a}} f(x) = f(a+0).$$

The left-hand and right-hand limits of a function at a point are generally called one-sided (or unilateral) limits, as opposed to two-sided (or bilateral) limit of a function at a point. If we study one-sided limits at the point $x = 0$, we usually simplify the notation and write $\lim_{x \to -0} f(x) = f(-0)$ for the left-hand limit, and $\lim_{x \to +0} f(x) = f(+0)$ for the right-hand limit.

Let us establish the relation between the one-sided limits and the limit of a function $f(x)$ at some point x_0. Let $f(x)$ be defined in a certain neighbourhood of the point x_0 except possibly for the point x_0 itself, that is, $f(x)$ is automatically defined on some intervals of the form (a, x_0) and (x_0, b). From the definitions themselves it follows that if $f(x)$ has a limit at the point x_0 and

$$\lim_{x \to x_0} f(x) = A, \tag{1}$$

then there also exist one-sided limits, and

$$f(x_0 + 0) = f(x_0 - 0) = A. \tag{2}$$

The converse statement is also true: if (2) holds, then (1) also holds. Thus, the limit of a function $f(x)$ at a point x_0 exists if and only if the following three conditions are satisfied: (a) the existence of the left-hand limit; (b) the existence of the right-hand limit; (c) the coincidence of the one-sided limits.

Example. Find the one-sided limits of the function

$$f(x) = \begin{cases} x+2, & x > 0, \\ x^3, & x \leqslant 0, \end{cases}$$

at the point $x = 0$. Determine whether this function has a limit as $x \to 0$.

△ The given function is defined throughout the number line (see Fig. 65). We evaluate the one-sided limits of this function at the

point $x = 0$:

$$f(+0) = \lim_{x \to +0} (x+2) = 2, \quad f(-0) \lim_{x \to -0} x^3 = 0.$$

Thus, $f(-0) \neq f(+0)$. Consequently, the given function has no limit at the point $x = 0$. ▲

5. On the Limit of a Function as $x \to \pm \infty$.

Definition 1. Let a function $f(x)$ be defined for all $x > b$, where b is some number. The number B is called the limit of $f(x)$ as $x \to +\infty$ if $\lim_{n \to \infty} f(x_n) = B$ for any sequence $\{x_n\}$ such that $x_n > b$ and $\lim_{n \to \infty} x_n = +\infty$. In this case we write

$$\lim_{x \to +\infty} f(x) = B.$$

Definition 2. Let a function $f(x)$ be defined for all $x < b$, where b is some number. The number C is called the limit of $f(x)$ as $x \to -\infty$ if $\lim_{n \to \infty} f(x_n) = C$ for any sequence $\{x_n\}$, such that $x_n < b$ and $\lim_{n \to \infty} x_n = -\infty$. In this case, we write:

$$\lim_{x \to -\infty} f(x) = C.$$

Example 1. Prove that

$$\lim_{x \to +\infty} \frac{x^3-2}{x^3+4} = 1.$$

△ Consider an arbitrary sequence $\{x_n\}$, such that

$$\lim_{n \to +\infty} x_n = +\infty.$$

Since the sequence $f(x_n) = \dfrac{x_n^3-2}{x_n^3+4} = 1 - \dfrac{6}{x_n^3+4}$, $n \in N$, converges to the definition,

$$\lim_{x \to +\infty} \frac{x^3-2}{x^3+4} = 1. \ ▲$$

It is easy to see that in the considered example we shall have $\lim_{x \to -\infty} \dfrac{x^3-2}{x^3+4} = 1$. But in the general case the function $f(x)$ behaves in a different case as $x \to +\infty$ and as $x \to -\infty$. Therefore, when investigating the properties of functions, we usually consider both $\lim_{x \to +\infty} f(x)$, and $\lim_{x \to -\infty} f(x)$.

Example 2. Evaluate the limits

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x-1} \text{ and } \lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x-1}.$$

△ Computing the limit as $x \to +\infty$, we assume that $x > 1$. Then $\dfrac{\sqrt{x^2+1}}{x-1} = \dfrac{\sqrt{1+\frac{1}{x^2}}}{1-\frac{1}{x}} \to 1$ as $x \to +\infty$. Computing the limit as $x \to -\infty$, we assume that $x < 0$. Then $x = -|x|$ and $\dfrac{\sqrt{x^2+1}}{x-1} = -\dfrac{\sqrt{1+\frac{1}{|x|^2}}}{1+\frac{1}{|x|}} \to -1$ as $x \to -\infty$. ▲

Example 3. Evaluate the limit

$$\lim_{x \to +\infty} \frac{8x^5+3x+9}{12-6x^2-x^5}.$$

△ We divide both the numerator and denominator by x^5. Then

$$\lim_{x \to +\infty} \frac{8x^5+3x+9}{12-6x^2-x^5} = \lim_{x \to +\infty} \frac{8+\frac{3}{x^4}+\frac{9}{x^5}}{\frac{12}{x^5}-\frac{6}{x^3}-1} = \frac{8}{-1} = -8. \blacktriangle$$

Example 4. Evaluate the limit $\lim\limits_{x \to +\infty} (\sqrt{x^2+1}-x)$.

△ Since $(\sqrt{x^2+1}-x) = \dfrac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} = \dfrac{1}{\sqrt{x^2+1}+x}$,

$$\lim_{x \to +\infty} (\sqrt{x^2+1}-x) = \lim_{x \to +\infty} \frac{1}{\sqrt{x^2+1}+x} = 0. \blacktriangle$$

6. Infinite Limits. Infinitely Small and Infinitely Large Functions. In addition to the above case of a finite limit of a function $f(x)$ as $x \to a$ (or $x \to \pm\infty$), the notion of infinite limit is used. For instance, the function $f(x) = \dfrac{1}{x^2}$, defined for all $x \neq 0$ (see 5.1, subsection 3) takes on arbitrary large values as $x \to 0$. In this case we say that at the point $x = 0$ the function has infinity as its limit and write: $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$.

Definition 1. If for any sequence of values of the argument $\{x_n\}$ is such that $x_n \neq a$ and $\lim\limits_{n \to \infty} x_n = a$, we have $\lim\limits_{n \to \infty} f(x_n) = \infty$, then we say that the limit of the function $f(x)$ at the point a is equal to infinity and write

$$\lim_{x \to a} f(x) = \infty.$$

Infinite limits for $x \to a \pm 0$ and for $x \to \pm\infty$ are defined in a similar way.

Definition 2. If $\lim\limits_{x \to a} f(x) = \infty$, then the function $f(x)$ is said to be infinitely large as $x \to a$.

Definition 3. If $\lim\limits_{x \to a} f(x) = 0$, then the function $f(x)$ is said to be infinitely small as $x \to a$.

Infinitely large and infinitely small functions as $x \to -\infty$, $x \to +\infty$ are defined in a similar manner.

The same as for sequences, the following assertion holds: if the function $f(x)$ is infinitely small as $x \to a$ and $f(x) \neq 0$ for $x \neq a$ from a certain neighbourhood of the point a, then the function $\frac{1}{f(x)}$ is infinitely large as $x \to a$.

The converse is also true: if the function $f(x)$ is infinitely large as $x \to a$, then the function $\frac{1}{f(x)}$ is infinitely small as $x \to a$.

For example, the function $f(x) = x$ is infinitely large as $x \to -\infty$ and $x \to +\infty$ and is infinitely small as $x \to 0$. The function $f(x) = \frac{1}{x}$ is infinitely small as $x \to \pm\infty$ and infinitely large as $x \to 0$.

Exercises

1. Prove the validity of the given equalities:

(a) $\lim\limits_{x \to 9} (2x) = 18$; (b) $\lim\limits_{x \to \frac{1}{2}} (2 - 8x) = -2$; (c) $\lim\limits_{x \to 1} (2x^2 - 3) = -1$.

2. Evaluate the following limits:

(a) $\lim\limits_{x \to 4} (x^4 - 2x + 5)$; (b) $\lim\limits_{x \to -1} \frac{x^3 + x^2 - 11}{8x^3 + 5}$;

(c) $\lim\limits_{x \to -2} \frac{x^3 + 8}{x + 2}$; (d) $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$;

(e) $\lim\limits_{x \to 1} \frac{x^2 + x - 2}{x - 1}$; (f) $\lim\limits_{x \to 1} \frac{x^2 - x}{\sqrt{x} - 1}$.

3. For the following functions clear up the existence of the limit at the points $x = -2$, $x = -1$, $x = 0$, $x = 1$ and $x = 2$.

(a) $f(x) = \begin{cases} 1, & x > 0, \\ -1, & x \leqslant 0; \end{cases}$ (b) $f(x) = \begin{cases} 2 - x, & x > 0, \\ x^2, & x \leqslant 0; \end{cases}$

(c) $f(x) = |x|$; (d) $f(x) = \frac{x + |x|}{x}$;

(e) $f(x) = \begin{cases} x^2 + x + 1, & x \geqslant 2, \\ x, & x < 2. \end{cases}$

4. Evaluate the left-hand and right-hand limits of the given functions:

(a) $\varphi(x)=\begin{cases}-x+1, & x\leqslant 1,\\ 2x-1, & x>1;\end{cases}$ at the point $x=1$,

(b) $\varphi(x)=\dfrac{1}{x+2^{\left(\frac{1}{x-3}\right)}}$ at the point $x=3$.

5. Evaluate the given limits:

(a) $\lim\limits_{x\to\pm\infty}\dfrac{1-3x}{3x+2}$; (b) $\lim\limits_{x\to\pm\infty}\dfrac{5x^2-6x-7}{9x^2+7x-8}$;

(c) $\lim\limits_{x\to\pm\infty}\dfrac{(x-1)^3}{x^3+6x^2-7x}$; (d) $\lim\limits_{x\to\pm\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})$;

(e) $\lim\limits_{x\to+\infty}(\sqrt{x^2+x-1}-\sqrt{x^2-x+1})$; (f) $\lim\limits_{x\to-\infty}\dfrac{6x^2+1}{2x^2+7}$;

(g) $\lim\limits_{x\to+\infty}\dfrac{4x+x+7}{x-3x-5}$; (h) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{7x^2+5}-x}{x+11}$.

5.7. Continuity of a Function

1. Continuity of a Function at a Point.

Definition 1. A function $f(x)$, $x\in(a, b)$ is called continuous at a point $x_0\in(a, b)$ if the limit of the function $f(x)$ at the point x_0 exists and is equal to the value of the function for this point:

$$\lim_{x\to x_0} f(x)=f(x_0).$$

According to the given definition, continuity of a function $f(x)$ at a point x_0 means the simultaneous feasibility of the following conditions:

(1) the function f must be defined at the point x_0;

(2) the function f must have a limit at the point x_0;

(3) the limit of the function f at the point x_0 coincides with the value of the function at this point.

For instance, the function $f(x)=x^2$ is defined throughout the number line and

$$\lim_{x\to 2} x^2=4.$$

Since $f(2)=4$, that is, the value of the function $f(x)=x^2$ at the point $x=2$ coincides with the limit as $x\to 2$, then, by definition, the function $f(x)=x^2$ is continuous at the point $x=2$.

Using the definition of the left-hand and right-hand limits of a function, we can define a left-side and right-side continuity of the

function, viz.: a function $f(x)$ is called continuous on the left at a point x_0 if

$$\lim_{x \to x_0 - 0} f(x) = f(x_0),$$

and continuous on the right at the point x_0 if

$$\lim_{x \to x_0 + 0} f(x) = f(x_0).$$

For example, the function

$$f(x) = \begin{cases} x + 2, & x > 0, \\ x^3, & x \leqslant 0, \end{cases}$$

whose graph is shown in Fig. 76 is continuous everywhere on the number line except for the point $x = 0$ at which it is continuous on the left since $f(0) = 0$ and $\lim\limits_{x \to -0} f(x) = \lim\limits_{x \to -0} x^3 = 0$.

It is possible to give another definition of continuity of a function using the notions of the increment in the argument and function. Let a function $f(x)$ be defined in the interval (a, b). The difference $x - x_0$, where $x \in (a, b)$ and $x_0 \in (a, b)$, is called the increment in the argument and is denoted by Δx, while the difference $f(x) - f(x_0)$ is termed the increment in the function corresponding to the given increment in the argument Δx and is denoted by Δf, that is,

$$\Delta x = x - x_0, \quad \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

In these notations the definition of continuity of a function at a point takes the following form:

Definition 2. A function $f(x)$, $x \in (a, b)$ is called continuous at a point $x_0 \in (a, b)$ if

$$\lim_{\Delta x \to 0} \Delta f = 0.$$

Hence it follows that the increment in a function continuous at x_0 is an infinitely small function as $\Delta x \to 0$.

Investigating the continuity of a function at a point, we of necessity use either Definition 1 or Definition 2.

Example 1. Prove that the function $\sqrt{x}$, $x \in [0, +\infty)$, is continuous at any point $x_0 > 0$.

△ Let us make use of Definition 1. We transform and estimate the absolute value of the difference for $x_0 > 0$:

$$0 \leqslant |\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leqslant \frac{|x - x_0|}{\sqrt{x_0}}. \tag{1}$$

Since $\lim\limits_{x \to x_0} |x - x_0| = 0$, and $1/\sqrt{x_0}$ is a constant function, we have

$$\lim_{x \to x_0} \frac{|x - x_0|}{\sqrt{x_0}} = 0. \tag{2}$$

According to the theorem on the limit of the intermediate function (see the preceding section), from formulas (1) and (2) it follows that

$$\lim_{x \to x_0} |\sqrt{x} - \sqrt{x_0}| = 0, \text{ or } \lim_{x \to x_0} (\sqrt{x} - \sqrt{x_0}) = 0,$$

and therefore

$$\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}.$$

Then, in accordance with Definition 1, the function $\sqrt{x}$ is continuous at any point $x_0 > 0$. At the point $x_0 = 0$ the function $\sqrt{x}$ is continuous on the right. ▲

Example 2. Prove that the function $f(x) = x^4$, $x \in \boldsymbol{R}$, is continuous at every point $x_0 \in \boldsymbol{R}$.

△ Here we use Definition 2. For any point $x_0 \in \boldsymbol{R}$ and any Δx we have:

$$\Delta f = (x_0 + \Delta x)^4 - x_0^4 = 4x_0^3 \cdot \Delta x + 6x_0^2 (\Delta x)^2 + 4x_0 (\Delta x)^3 + (\Delta x)^4.$$

Using the theorems on the limit of the sum and product of functions, we find:

$$\lim_{\Delta x \to 0} \Delta f = 0,$$

consequently, the function $f(x) = x^4$ is continuous at any point $x_0 \in \boldsymbol{R}$. ▲

Example 3. Investigate the function $y = \tan x$, $x \neq \pm \frac{\pi}{2} + \pi k$, $k \in \boldsymbol{Z}$, for continuity at the point $x_0 = \frac{\pi}{2}$ (Fig. 82).

△ The function under consideration is not continuous at the point $x_0 = \frac{\pi}{2}$ since it is not defined at this point. ▲

Example 4. Investigate the function

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \in \boldsymbol{R}, \quad x \neq 0, \\ 0, & x = 0, \end{cases}$$

for continuity at the point $x_0 = 0$.

△ We have $f(x_0) = 0$. Since $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{1}{x^2} = \infty$, the function is not continuous at the point $x_0 = 0$. ▲

2. The Properties of Functions Continuous at a Point.

Theorem 5.7.1. *The sum (difference) of a finite number of functions continuous at a point x_0 is a function continuous at this point.*

Theorem 5.7.2. *The product of a finite number of functions continuous at a point x_0 is a function continuous at this point.*

Theorem 5.7.3. *The ratio of two functions continuous at a point x_0 is a function continuous at this point if the value of the function in the denominator is different from zero at the point x_0.*

The validity of these theorems follows from the corresponding theorems on the limits of functions and Definition 1 of continuity.

Example 1. Prove that the function $y = x^n$, $x \in R$, $n \in N$, is continuous throughout the number line.

△ The function

$$f(x) = x^n = x \cdot x \cdot \ldots \cdot x$$

is the product of n factors. Taking into account the continuity of x, we get from Theorem 5.7.2 that the function $f(x) = x^n$ is continuous everywhere on the number line. ▲

Example 2. Prove that the function $f(x) = cx^n$ (c constant) is continuous throughout the number line.

△ According to Example 1, the function x^n is continuous throughout the number line. The same is true with respect to the constant function $y = c$, $x \in R$. Then, Theorem 5.7.2 implies that the function cx^n is continuous everywhere on the number line. ▲

Example 3. Prove that a polynomial is a function continuous throughout the number line.

△ Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$. The functions $f_0(x) = a_0x^n$, $f_1(x) = a_1x^{n-1}$, $\ldots$, $f_{n-1}(x) = a_{n-1}x$, $f_n(x) = a_n$ represent functions continuous throughout the number line (see Example 2). Consequently, considering the polynomial as the sum of these functions, by Theorem 5.7.1, it is a function continuous on R. ▲

Example 4. Prove that any rational function is continuous at any point of its domain of definition.

△ A rational function has the form $f(x) = \frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomials. Since, according to Example 3, $P_n(x)$ and $Q_m(x)$ are continuous everywhere on the number line, and the polynomial $Q_m(x)$ is nonzero in the domain of definition of the function $f(x)$, by Theorem 5.7.3 the function $f(x)$ is continuous at any point of its domain. ▲

3. Continuity of a Function on a Set. If a function is continuous at each point of the interval (a, b), then it is said to be continuous on this interval. A function is said to be continuous on the closed interval $[a, b]$ if it is continuous on the interval (a, b) and, in addition, continuous on the left at the point b and continuous on the

right at the point a. Thus, at the points a and b (that is, at the end points of the interval $[a, b]$) only one-sided continuity of the function is required. For instance, the function $y = [x]$ (see Fig. 6) is continuous on every interval of the form $(n, n+1)$, $n \in Z$, and is continuous on the right at points $x = n \in Z$. The function

$$y = \begin{cases} x+2, & x > 0, \\ x^3, & x \leqslant 0 \end{cases}$$

(see Fig. 65) is continuous on the infinite intervals $(0, +\infty)$ and $(-\infty, 0]$. It is continuous on the left at the point $x = 0$.

Functions continuous on a closed interval possess a number of important properties. Given below without proof are two theorems characterizing these properties.

Theorem 5.7.4. *If a function f is continuous on a closed interval $[a, b]$ and takes on values of different sign at its end points, then there is at least one point inside this interval at which the given function vanishes.*

Theorem 5.7.5. *If a function f is continuous on a closed interval $[a, b]$, then among the values assumed by the function on this interval there are the least and greatest values, and the function takes on all the values between the greatest and least values.*

Theorem 5.7.4 is used for finding the roots of the equation

$$f(x) = 0. \tag{1}$$

If it is possible to indicate a closed interval $[a, b]$ such that $f(x)$ is continuous on $[a, b]$, and, for instance, $f(a) < 0$ and $f(b) > 0$, then, according to Theorem 5.7.4, we may assert that there is at least one root of equation (1) on the interval (a, b).

Example. Prove that the equation $f(x) = x - \cos x = 0$ has at least one root on the closed interval $\left[0, \frac{\pi}{2}\right]$.

△ Since $f(0) = -\cos(0) = -1 < 0$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$, and the function $f(x)$ is continuous on this interval (continuity of the function $\cos x$ on R will be proved below), then, by Theorem 5.7.4, there is at least one root of the equation $x - \cos x = 0$ on the interval $\left(0, \frac{\pi}{2}\right)$. ▲

4. Points of Discontinuity and Their Classification. If a function $f(x)$ is continuous at a point x_0, then the point x_0 is called the point of continuity of the function $f(x)$. Otherwise, that is, when the limit of the function $f(x)$ at the point x_0 does not exist or exists but is not equal to $f(x_0)$, the function is said to be discontinuous at the point x_0, the latter being called the point of discontinuity of the function

$f(x)$. In particular, if $f(x)$ is defined for all points of the interval (a, b) except for the point $x_0 \in (a, b)$, then x_0 is also a point of discontinuity of the function $f(x)$. Points of discontinuity are customarily classified according to the following definition.

Definition. A point of discontinuity of a function is called a point of discontinuity of the first kind if the function has finite limits on the right and on the left at this point. In all other cases, a point of discontinuity is termed a point of discontinuity of the second kind.

According to this definition, at the point of discontinuity of the first kind x_0 the function $f(x)$ undergoes a finite jump equal to $f(x_0 + 0) - f(x_0 - 0)$; and if x_0 is a point of discontinuity of the second kind of the function $f(x)$, then at least one of its one-sided limits does not exist at the point x_0 or is equal to infinity.

For example, the function $y = [x]$ (see Fig. 66) has discontinuities of the first kind at the points $x = n$, $n \in \boldsymbol{Z}$, and the function

$$x = \begin{cases} x+2, & x > 0, \\ x^3, & x \leqslant 0 \end{cases}$$

(see Fig. 65) suffers a discontinuity of the first kind at the point $x = 0$. The jump of the function is equal to $f(+0) - f(-0) = 2$. The function $y = \tan x$ (see Fig. 82) undergoes discontinuities of the second kind at points $x = \pm \frac{\pi}{2} + \pi k$, $k \in \boldsymbol{Z}$. The rational function $y = \frac{2x-1}{x^2-2x-3}$ (see Fig. 90) undergoes discontinuities of the second kind at the points $x = -1$ and $x = 3$.

Exercises

1. Investigate the following functions for continuity:

(a) $f(x) = 4x + 1$ at the points $x = 1$ and $x = -1$;

(b) $f(x) = \begin{cases} x^2 - 1, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$ at the points $x = 0$, $x = -1$, $x = 1$;

(c) $f(x) = \begin{cases} x, & x \geqslant 0, \\ 1 + x^2, & x < 0, \end{cases}$ at the points $x = -2$, $x = 0$, $x = 1$;

(d) $f(x) = x - |x|$ at the points $x = -2$, $x = 0$, $x = 5$;

(e) $f(x) = \begin{cases} 2|x|, & |x| \leqslant 1, \\ 1, & |x| > 1, \end{cases}$ at the points $x = -1$, $x = 0$, $x = 3$:

(f) $f(x) = 17x^4 - 11x^3 + 2x^2 - 7x + 8$ at the points $x = -1$, 0, 1;

(g) $f(x) = \frac{7x^2 + x - 9}{x + 2}$ at the points $x = -2$, 0, 2.

2. Find the points of discontinuity and determine their kind for the following functions:

(a) $f(x) = \begin{cases} 2-x^2, & x \geqslant 0, \\ x^2-2, & x<0, \end{cases}$

(b) $f(x) = x-[x]$, where $[x]$ is the integral part of x;

(c) $f(x) = \cot x$; (d) $f(x) = \dfrac{3x}{x^2-4x+4}$.

3. Determine whether the given functions are continuous on the indicated intervals:

(a) $f(x) = 5x^7-6x^5+4$ on $[-100, 378]$;

(b) $f(x) = \tan x$ on $\left[0, \frac{\pi}{2}\right]$;

(c) $f(x) = \tan x$ on $\left[0, \frac{\pi}{4}\right]$;

(d) $f(x) = x-[x]$ on $[0, 1]$;

(e) $f(x) = x-[x]$ on $\left[\frac{1}{3}, \frac{7}{8}\right]$.

5.8. Continuity of Elementary Functions. Remarkable Limits

1. Continuity of Simplest Elementary Functions. When proving the continuity of the simplest elementary functions listed in Table 1 (see 5.2), we shall be guided by the following theorem which is given without proof.

Theorem 5.8.1. *Let there be given a strictly monotonic continuous function $y = f(x)$ on an open interval (a, b), and let $\alpha = \lim\limits_{x \to a} f(x)$ and $\beta = \lim\limits_{x \to b} f(x)$ (finite or infinite) exist. Then this function has a strictly monotonic and continuous inverse function $x = f^{-1}(y)$ on the interval (α, β) (or (β, α) if $\beta < \alpha$).*

For the constant function $f(x) = C$, $x \in \boldsymbol{R}$, we easily find $\Delta f = f(x) - f(x_0) = C - C = 0$, and then $\lim\limits_{\Delta x \to 0} \Delta f = 0$. Hence, a constant function is continuous throughout the number line.

Consider now the exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$, $x \in \boldsymbol{R}$. Let us prove that an exponential function is continuous on the entire number line $\boldsymbol{R}$.

□ Consider first the case $a > 1$. Let $\{x_n\}$ be a sequence converging to a certain point x_0 of the number line. Let us show that the sequence $y_n = a^{x_n}$, $n \in \boldsymbol{N}$, converges to $y_0 = a^{x_0}$. Let us take some $\varepsilon > 0$. Without loss of generality, we shall assume that $\varepsilon < y_0$. Then from

the properties of logarithms it follows that

$$\log_a (y_0 - \varepsilon) < x_0 < \log_a (y_0 + \varepsilon).$$

Since $\lim\limits_{n\to\infty} x_n = x_0$, there is N_1 such that $\log_a (y_0 - \varepsilon) < x_n$ for all $n > N_1$, and there is N_2 such that $x_n < \log_a (y_0 + \varepsilon)$ for all $n > N_2$. Then for any $x > N = \max \{N_1, N_2\}$ the following inequality is valid:

$$\log_a (y_0 - \varepsilon) < x_n < \log_a (y_0 + \varepsilon).$$

Hence it follows that

$$y_0 - \varepsilon < a^{x_n} < y_0 + \varepsilon$$

for any $n > N$. Since ε is an arbitrary positive number, consequently, we have proved that $\lim\limits_{n\to\infty} a^{x_n} = y_0$. And since $\{x_n\}$ is an arbitrary sequence converging to x_0, we have

$$\lim_{x\to x_0} a^x = a^{x_0}.$$

If $0 < a < 1$, then $a^{-1} > 1$, and, consequently,

$$\lim_{x\to x_0} a^x = \frac{1}{\lim\limits_{x\to x_0} (a^{-1})^x} = \frac{1}{(a^{-1})^{x_0}} = a^{x_0}. \quad \blacksquare$$

Since the logarithmic function

$$y = \log_a x, \quad x \in \boldsymbol{R}_+, \quad a > 0, \quad a \neq 1,$$

is the inverse of the exponential function a^x, strictly monotonic and continuous on the set $\boldsymbol{R}$, then, by Theorem 5.8.1, it is continuous on the set $\boldsymbol{R}_+ = (0, +\infty)$ (since it is easy to prove that $\lim\limits_{x\to-\infty} a^x = 0$, $\lim\limits_{x\to+\infty} a^x = +\infty$ for $a > 1$ and $\lim\limits_{x\to-\infty} a^x = +\infty$, $\lim\limits_{x\to+\infty} a^x = 0$ for $0 < a < 1$), that is, in the entire domain of definition.

Let us now consider the power function. When proving the continuity of this function, we shall use the following theorem.

Theorem 5.8.2. *If the function $y = \varphi(x)$ is continuous at a point x_0, and the function $z = f(y)$ is continuous at the point $y_0 = \varphi(x_0)$, then the composite function $z = f(\varphi(x))$ is continuous at the point x_0.*

□ The function $y = \varphi(x)$ is continuous at the point x_0, therefore for any sequence $\{x_n\}$ converging to x_0, the sequence

$$y_n = \varphi(x_n), \quad n \in \boldsymbol{N}, \tag{1}$$

converges to $y_0 = \varphi(x_0)$. The function $z = f(y)$ is continuous at the point y_0, therefore for sequence (1) the sequence

$$z_n = f(y_n), \quad n \in \boldsymbol{N},$$

converges to $z_0 = f(y_0)$. Combining both assertions, we find that for any sequence $\{x_n\}$ converging to x_0 the sequence $z_n = f(y_n) = f(\varphi(x_n))$, $n \in N$, converges to $z_0 = f(y_0) = f(\varphi(x_0))$. And this just means that the composite function $z = f(\varphi(x))$ is continuous at the point x_0. ■

In other words, if the conditions of the theorem are satisfied, then

$$\lim_{x \to x_0} f(\varphi(x)) = \lim_{y \to y_0} f(y). \tag{2}$$

Formula (2) is called the formula for changing the variable under the sign of limit.

From Theorem 5.8.2 it also follows that

$$\lim_{x \to x_0} f(\varphi(x)) = f(\varphi(x_0)) = f(y_0) = f\left(\lim_{x \to x_0} \varphi(x)\right),$$

or, shorter,

$$\lim_{x \to x_0} f(\varphi(x)) = f\left(\lim_{x \to x_0} \varphi(x)\right). \tag{3}$$

Formula (3) means that it is possible to interchange the signs of limit and a continuous function.

Since the power function $y = x^\alpha$, $x \in \boldsymbol{R}_+$ can be represented in the form

$$y = x^\alpha = a^{\alpha \log_a x}, \quad x \in \boldsymbol{R}_+,$$

that is, in the form of a superposition of continuous functions (exponential and logarithmic), by Theorem 5.8.2, the power function is continuous at any point $x_0 \in \boldsymbol{R}_+$.

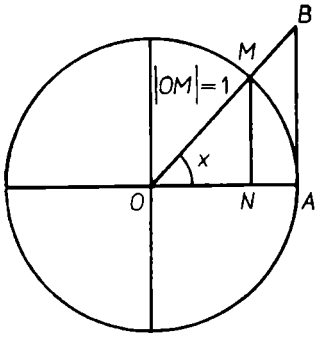

Fig. 96

To prove the continuity of trigonometric functions, we shall need the following theorem:

Theorem 5.8.3. *For any* $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *the following inequalities are valid*

$$|\sin x| \leqslant |x| \leqslant |\tan x|. \tag{4}$$

□ For $x = 0$ formula (4) is valid since in this case $\sin x = x = \tan x = 0$. Let now $0 < x < \frac{\pi}{2}$. Consider a circle of unit radius with centre at point O and the point A on this circle (Fig. 96). We shall count off arcs from the point A anticlockwise. Let M be a point of the circle located in the first quadrant, and x the length of the arc AM, $0 < x < \frac{\pi}{2}$ (x radian measure of the angle MOA), and N the foot of the perpendicular dropped from M on OA, B the point of intersection of the perpendicular to OA erected at the point A

and the extension of the line segment OM. Then, by definition, $MN = \sin x$, $ON = \cos x$, and $AB = \tan x$. Since the triangle OMA is contained in the sector OMA which in turn is enclosed in the triangle OBA, and the areas of these figures are equal to $\frac{1}{2}\sin x$, $\frac{x}{2}$, and $\frac{1}{2}\tan x$, respectively, the following inequalities are valid: $0 < \sin x < x < \tan x$ or

$$|\sin x| < |x| < |\tan x|. \tag{5}$$

By virtue of oddness of the functions x, $\sin x$, and $\tan x$, formula (5) also holds true for $-\frac{\pi}{2} < x < 0$. Thus, we have proved that for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have the inequalities $|\sin x| \leqslant |x| \leqslant |\tan x|$. ■

It is clear that the inequality

$$|\sin x| \leqslant |x| \tag{6}$$

is valid for any $x \in \boldsymbol{R}$, since if $|x| \geqslant \frac{\pi}{2}$, then $|\sin x| \leqslant 1 < \frac{\pi}{2} \leqslant |x|$.

With the help of inequality (6), formula

$$\Delta y = \sin x - \sin x_0 = 2 \sin \frac{x - x_0}{2} \cos \frac{x + x_0}{2}$$

and inequality $\left|\cos \frac{x + x_0}{2}\right| \leqslant 1$, we find that $0 \leqslant |\Delta y| = 2 \cdot \left|\sin \frac{x - x_0}{2}\right| \times \left|\cos \frac{x + x_0}{2}\right| \leqslant 2 \cdot \left|\frac{x - x_0}{2}\right| \cdot 1 = |x - x_0|$. Since $\lim\limits_{x \to x_0} |x - x_0| = 0$, by the theorem on the limit of an intermediate function (see 5.6, subsection 3), we have:

$$\lim_{x \to x_0} (\sin x - \sin x_0) = 0, \text{ that is, } \sin x \to \sin x_0$$

as $x \to x_0$, and this just means that the function $\sin x$ is continuous at an arbitrary point $x_0 \in \boldsymbol{R}$.

The continuity of the function $y = \cos x$ follows from Theorem 5.8.2 and the continuity of the sine and a linear function throughout the number line $\boldsymbol{R}$ if it is written in the form $y = \sin\left(\frac{\pi}{2} - x\right)$.

The trigonometric functions $\tan x$ and $\cot x$ are continuous in the domain of their definition. This assertion follows from the theorem on the continuity of the quotient of continuous functions.

The continuity of the inverse trigonometric functions $\arcsin x$, $x \in [-1, 1]$, $\arccos x$, $x \in [-1, 1]$, $\arctan x$, $x \in \boldsymbol{R}$, and $\operatorname{arccot} x$,

$x \in R$, at any point of the domain of definition follows from Theorem 5.8.1.

Thus, all simplest elementary functions are continuous in the domain of definition. An arbitrary elementary function is obtained as a result of a finite number of arithmetical operations on the simplest elementary functions, and also by means of superposition of these functions. On the basis of Theorem 5.8.2 and Theorems 5.7.1, 5.7.2, and 5.7.3 from subsection 2 of the preceding section, we conclude that any elementary function is continuous in the domain of its definition.

There is no need now to use the definition of limit each time when evaluating limits. It suffices to take advantage of the continuity of elementary functions and the basic theorems on limits, the theorem on the limit of a composite function included.

Example 1. Evaluate the limit $\lim\limits_{x \to 2} \dfrac{3x-8}{4x+2}$.

△ By virtue of the theorem on the limit of a quotient of two functions and the continuity property of elementary functions, we have:

$$\lim_{x \to 2} \frac{3x-8}{4x+2} = \frac{\lim\limits_{x \to 2} (3x-8)}{\lim\limits_{x \to 2} (4x+2)} = \frac{3 \cdot 2 - 8}{4 \cdot 2 + 2} = -\frac{1}{5}. \blacktriangle$$

Example 2. Evaluate the limit $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{x+1}-1}{x}$.

△ Since the limits of both the numerator and denominator are simultaneously equal to zero, it is impossible to use directly the theorem on the limit of a quotient. Let us introduce a new variable $y = \sqrt[3]{x+1}$, which is a continuous function of x. Then, by the formula for changing the variable under the sign of limit, we have

$$\lim_{x \to 0} \frac{\sqrt[3]{x+1}-1}{x} = \lim_{y \to 1} \frac{y-1}{y^3-1}.$$

The last limit is readily computed after identity transformations:

$$\frac{y-1}{y^3-1} = \frac{y-1}{(y-1)(y^2+y+1)} = \frac{1}{y^2+y+1}.$$

Finally,

$$\lim_{x \to 0} \frac{\sqrt[3]{x+1}-1}{x} = \lim_{y \to 1} \frac{y-1}{y^3-1} = \frac{1}{\lim\limits_{y \to 1} (y^2+y+1)} = \frac{1}{3}. \blacktriangle$$

Example 3. Evaluate the limit

$$\lim_{x \to \pi} (\sin 2x + e^x \arctan x).$$

△ By virtue of the continuity of elementary functions, we have:

$$\lim_{x\to\pi} \sin 2x = \sin 2\pi = 0, \quad \lim_{x\to\pi} e^x = e^\pi,$$

$$\lim_{x\to\pi} \arctan x = \arctan \pi.$$

By the theorem on the limit of the sum and product of functions, we find:

$$\lim_{x\to\pi} (\sin 2x + e^x \arctan x) = \lim_{x\to\pi} \sin 2x + \lim_{x\to\pi} e^x \cdot \lim_{x\to\pi} \arctan x$$

$$= e^\pi \cdot \arctan \pi. \ ▲$$

Example 4. Evaluate the limit

$$\lim_{x\to 0} \arctan (\cos x).$$

△ Since we may interchange the signs of the limit and a continuous function, we have:

$$\lim_{x\to 0} \arctan (\cos x) = \arctan (\lim_{x\to 0} \cos x).$$

By virtue of the continuity of elementary functions, we have $\lim_{x\to 0} \cos x = \cos 0 = 1$. Then

$$\lim_{x\to 0} \arctan (\cos x) = \arctan (1) = \frac{\pi}{4}. \ ▲$$

2. Remarkable Limits. The limits

$$\lim_{x\to 0} \frac{\sin x}{x} \text{ and } \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

are customarily called the first and second remarkable limits owing to their importance in further applications.

Theorem 5.8.4. *The limit* $\lim_{x\to 0} \frac{\sin x}{x}$ *exists and is equal to unity.*

□ It was proved in the preceding subsection that the inequality

$$|\sin x| < |x| < |\tan x|$$

is valid for $0 < |x| < \frac{\pi}{2}$. Dividing this inequality by the positive quantity $|\sin x|$, we get the inequality

$$1 < \left|\frac{x}{\sin x}\right| < \frac{1}{|\cos x|}.$$

This inequality is equivalent to the inequality

$$|\cos x| < \left|\frac{\sin x}{x}\right| < 1,$$

which by virtue of the evenness of the functions $\cos x$ and $\frac{\sin x}{x}$, can be written in the form

$$\cos x < \frac{\sin x}{x} < 1$$

if $0 < |x| < \frac{\pi}{2}$. Since $\lim\limits_{x \to 0} \cos x = 1$, by the theorem on the limit of intermediate function, we find:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \blacksquare$$

Theorem 5.8.5. *The limit* $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ *exists and is equal to the number* e.

□ In subsection 6 of 5.5 we proved that the limit $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ exists, and its value was denoted by the letter e. Since $[x] \leqslant x \leqslant [x] + 1$, where $[x]$ is the integral part of x, we have the following inequality:

$$\left(1 + \frac{1}{[x]+1}\right)^{[x]} \leqslant \left(1 + \frac{1}{x}\right)^x \leqslant \left(1 + \frac{1}{[x]}\right)^{[x]+1} \tag{1}$$

for any $x > 1$. It is obvious that

$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]+1}\right)^{[x]} = \lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right)^n = e,$$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{[x]}\right)^{[x]+1} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \cdot \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = e,$$

since, as $x \to +\infty$, the function $[x]$ takes on positive integral values $n = [x]$ which also tend to infinity. Then from inequality (1), by the theorem on the limit of an intermediate function, we find:

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e. \tag{2}$$

Let us consider now the case when $x \to -\infty$. We introduce a new variable $y = -1 - x$ which tends to $+\infty$. Making use of the formula for changing the variable under the sign of limit, we find:

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{y \to +\infty} \left(\frac{-y}{-1-y}\right)^{-1-y} = \lim_{y \to +\infty} \left(\frac{1+y}{y}\right)^{y+1}$$

$$= \lim_{y \to +\infty} \left(1 + \frac{1}{y}\right)^y \cdot \lim_{y \to +\infty} \left(1 + \frac{1}{y}\right) = e. \tag{3}$$

Combining results (2) and (3) we find:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e. \blacksquare$$

Example 1. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$.

△ We set $y = 2x$. Then, by the formula for changing the variable under the sign of limit, we have:

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{y\to 0}\frac{2\sin y}{y}=2\lim_{y\to 0}\frac{\sin y}{y}=2\cdot 1=2. \ ▲$$

Example 2. Evaluate the limit

$$\lim_{t\to 0}\frac{e^t-1}{t}.$$

△ We set $e^t-1=\dfrac{1}{x}$. Then, by the formula for changing the variable under the sign of limit, we have:

$$\lim_{t\to 0}\frac{e^t-1}{t}=\lim_{x\to\infty}\frac{1}{x\ln\left(1+\frac{1}{x}\right)}=\frac{1}{\lim\limits_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x}.$$

Interchanging the signs of the limit and continuous function, we find:

$$\lim_{t\to 0}\frac{e^t-1}{t}=\frac{1}{\ln\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x}=\frac{1}{\ln e}=1. \ ▲$$

Exercises

Evaluate the following limits:

(a) $\lim\limits_{x\to 1}\dfrac{\sqrt{5-x}-2}{\sqrt{2-x}-1}$; (b) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{9x^2+2}-x}{4x+11}$;

(c) $\lim\limits_{x\to 2}(x^2+7x-9)$; (d) $\lim\limits_{x\to 1}\dfrac{5x+8}{4x-7}$;

(e) $\lim\limits_{x\to 0}\dfrac{3x+2x^2}{x^2-x+1}$; (f) $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x^2-5x+6}$;

(g) $\lim\limits_{x\to 1}\dfrac{x^6-1}{x^3-1}$; (h) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$;

(i) $\lim\limits_{x\to 8}\dfrac{\sqrt[3]{x}-2}{x-8}$; (j) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{5-x}-\sqrt{5+x}}$;

(k) $\lim\limits_{x\to 3}\dfrac{\sqrt{2x+10}-4}{x-3}$; (l) $\lim\limits_{x\to\infty}\dfrac{x-\sin x}{1-2x}$;

(m) $\lim\limits_{x\to -1}\dfrac{x^3+1}{\sin(x+1)}$; (n) $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{\sqrt{x}-1}$; (o) $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{\sin 5x}$.

Chapter 6

ELEMENTS OF DIFFERENTIAL CALCULUS: FUNCTIONS OF ONE VARIABLE

6.1. Derivatives

1. Problems Leading to the Concept of Derivative. Consider the motion of a material point M along the axis Ox (Fig. 97). For the reference point (point O) we take the position of the material point at the instant of time $t = 0$. Let at time instant t the coordinate of the moving point x be equal to $f(t)$, that is, the coordinate x of the material point is a function of time:

$$x = f(t), \quad t \in [0, T]. \tag{1}$$

This function is called the law of motion of the point M. Sometimes it is written in the form

$$x = x(t), \quad t \in [0, T].$$

As is known from physics, the law of uniform rectilinear motion is given by the formula

$$x = vt. \tag{2}$$

In practice, trains, automobiles, etc. perform uniform rectilinear motion only in certain intervals of their path, however, in the gen-

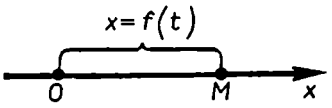

Fig. 97

eral case their motion is nonuniform. Performing nonuniform rectilinear motion which is described by function (1) differing from function (2), a material point may perform different displacements, both in magnitude and direction, during equal intervals of time. Consequently, nonuniform motion (in contrast to uniform motion) cannot be completely characterized only by indicating the path travelled by the point during a certain interval of time. Therefore, to describe nonuniform motion of a point, the concept of average velocity v_{av} is introduced which depends on the choice of time instants t_1 and t_0:

$$v_{av}(t_1, t_0) = \frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

Let us illustrate this by an example. As is known from physics, free fall of bodies in the gravitational field of the Earth is a nonuni-

form motion described by the law $x = \frac{gt^2}{2}$, where g is acceleration due to gravity. The average velocity during the first second of body's motion, that is, during the time interval from the instant $t_0 = 0$ till the instant $t_1 = 1$ is equal to

$$v_{av}(1, 0) = \frac{g}{2},$$

whereas for the second second of free fall ($t_1 = 2$, $t_0 = 1$) it increases by a factor of three:

$$v_{av}(2, 1) = \left(\frac{gt_1^2}{2} - \frac{gt_0^2}{2}\right)\Big/(t_1 - t_0) = \frac{g}{2}(t_1 + t_0) = \frac{3g}{2}.$$

Average velocity cannot completely characterize nonuniform motion. Therefore we introduce the so-called instantaneous velocity. Obviously, the less the duration $t_1 - t_0$ of a time interval, the more complete is the characteristic of motion given by average velocity. The instantaneous velocity of rectilinear motion at a given instant t_0 is defined as the limit (provided it exists) to which the average velocity tends during the time interval $t_1 - t_0$ as t_1 tends to t_0, that is,

$$v(t_0) = \lim_{t_1 \to t_0} v_{av}(t_1, t_0) = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

Example 1. Find the instantaneous velocity of a freely falling body in the gravitational field of the Earth at the instant t_0.

△ Since the law of free fall of a body is $x = \frac{gt^2}{2}$, we have

$$v(t_0) = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0} = \lim_{t_1 \to t_0} \frac{\frac{gt_1^2}{2} - \frac{gt_0^2}{2}}{t_1 - t_0}$$

$$= \frac{g}{2} \lim_{t_1 \to t_0} \frac{t_1^2 - t_0^2}{t_1 - t_0} = \frac{g}{2} \lim_{t_1 \to t_0} (t_1 + t_0) = gt_0.$$

Thus, a freely falling body moves with the velocity $v(t) = gt$. ▲

Example 2. When switched on, the lift moves according to the law $x = 2t^2 + 3t + 1$. Find the instantaneous velocity at the instant t_0.

△ By the definition of instantaneous velocity, we get:

$$v(t_0) = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0} = \lim_{t_1 \to t_0} \frac{(2t_1^2 + 3t_1 + 1) - (2t_0^2 + 3t_0 + 1)}{t_1 - t_0}$$

$$= \lim_{t_1 \to t_0} \frac{2(t_1^2 - t_0^2) + 3(t_1 - t_0)}{t_1 - t_0} = \lim_{t_1 \to t_0} [2(t_1 + t_0) + 3] = 4t_0 + 3.$$

Thus, on being switched on, the lift moves with the velocity $v(t) = 4t + 3$. ▲

Let us now consider the problem on line density of a bar. Let there be given a bar of length L (Fig. 98) arranged along the axis Ox, its left-hand end being brought into coincidence with the reference point (point O). To each section of the bar of length x, $0 \leqslant x \leqslant L$, measured from its left-hand end, a mass m corresponds, that is, the mass of the section of the bar is a function of its length: $m = m(x)$,

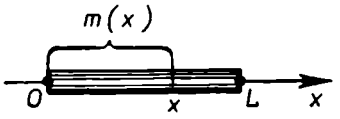

Fig. 98

$x \in [0, L]$. A bar is said to be homogeneous if any two of its sections of equal length have the same mass. In this case the ratio of the mass of any section of the bar to its length is a constant which is called the line density of the bar:

$$\rho = \frac{m(x_1) - m(x_0)}{x_1 - x_0}. \tag{3}$$

Setting $x_0 = 0$ and $x_1 = x$ in formula (3), we find that for a homogeneous bar

$$m(x) = \rho x, \tag{4}$$

so that ρ characterizes the rate of increase in the mass of a variable section depending of its length x.

In practice, we come across nonhomogeneous bars in which two different sections of the same length have, generally speaking, different masses, and the function $m(x)$ characterizing them differs from function (4). For them, according to formula (3), we determine the average line density ρ_{av} which depends on the coordinates x_1 and x_0:

$$\rho_{av}(x_1, x_0) = \frac{m(x_1) - m(x_0)}{x_1 - x_0}.$$

In the case of a nonhomogeneous bar the average line density ρ_{av} cannot completely characterize the rate of change in the mass of the bar. Therefore, for nonhomogeneous bars we introduce the notion of line density $\rho(x_0)$ at a given point x_0. This is done in the following way:

$$\rho(x_0) = \lim_{x_1 \to x_0} \rho_{av}(x_1, x_0) = \lim_{x_1 \to x_0} \frac{m(x_1) - m(x_0)}{x_1 - x_0}.$$

Example 3. Given a thin bar of length L. The mass of the portion of the bar measured from its left-hand end (see Fig. 98) is described by the function

$$m(x) = Ax^2 + Bx,$$

where A and B are constants. Find the line density of the bar at the point $x_0 \in [0, L]$.

△ By the definition of line density at a point, we get:

$$\rho(x_0) = \lim_{x_1 \to x_0} \frac{m(x_1) - m(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{Ax_1^2 + Bx_1 - Ax_0^2 - Bx_0}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{A(x_1^2 - x_0^2) + B(x_1 - x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} [A(x_1 + x_0) + B] = 2Ax_0 + B.$$

For the given nonhomogeneous bar, the line density at a point is defined by the following function of x: $\rho(x) = 2Ax + B$. ▲

2. The Derivative of a Function. In the above discussed problems various physical quantities were introduced with the help of a certain limit of the same form. Therefore it is reasonable to consider such a limit for a function in the general case.

Definition. Let there be given a function $f(x)$, $x \in (a, b)$ and let x_0 be a certain point belonging to the interval (a, b). The limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is called the derivative of the function $f(x)$ at the point x_0 and is denoted by $f'(x_0)$. Thus, by definition,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{1}$$

A function having a derivative at some point is called differentiable at this point. A function $f(x)$, $x \in (a, b)$ having a derivative at each point of the interval (a, b) is said to be differentiable on this interval. The operation of finding the derivative of a given function is called differentiation and is denoted with the help of a dash.

With the increment in the argument $\Delta x = x - x_0$ and increment in the function $\Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$ introduced, the derivative of the function $f(x)$, at the point x_0 according to (1) is written in the form

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{2}$$

Instead of a dash, the derivative is frequently denoted by the symbol $\frac{df}{dx}$ (read: "df over dx").

Since x_0 is an arbitrary value of the argument, we shall denote it simply by x. Then formula (2) takes the form:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{3}$$

Returning to the problems considered in the preceding subsection, we may now say that the required values of the instantaneous velocity of motion $v(t)$ and the line density of a nonhomogeneous bar $\rho(x)$ at a point are derivatives of the corresponding functions:

$$v(t) = x'(t), \quad \rho(x) = m'(x).$$

3. Physical and Geometrical Meaning of the Derivative. We approached the concept of derivative as a result of considering the velocity of a moving body (material point), that is, proceeding from mechanical considerations. It turned out that in problems of other physical nature, where we do not speak of motion introduced with the help of the derivative, nevertheless we can impart to the considered concepts the meaning of some rate (instead of velocity). Indeed, for a nonhomogeneous (nonuniform) bar the line density at a point is equal to the rate of increase in the mass of a variable portion of the bar with a change in its length. There are many examples of problems in which the quantities introduced by the derivative have the meaning of the rate of change of certain functions: the problem on heat capacity of a body, the problem on finding the angular velocity of a rotating body, and so forth.

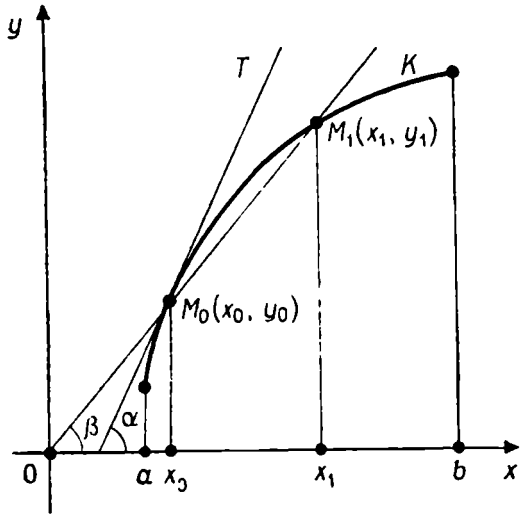

Fig. 99

Now, we are going to give a geometrical interpretation of the derivative which is no less important. Let the curve K be the graph of a continuous function $y = f(x)$, $x \in (a, b)$ (Fig. 99). We take the points $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$ on the curve K and draw a secant M_0M_1. Its slope $k = \tan\beta$ is equal to

$$\tan\beta = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Let now $x_1 \to x_0$, that is, the abscissa of the point M_1 approaches the abscissa of the point M_0, and, consequently, the point M_1 tends to the point M_0 without leaving the curve K. Under these conditions the secant M_0M_1, generally speaking, changes its position rotating about the point M_0, this means that the angle β varies.

If the function $f(x)$ is differentiable at the point x_0, then there exists the limit

$$\lim_{x_1 \to x_0} \tan\beta = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \tan\alpha = f'(x_0),$$

and, consequently, there exists a straight line M_0T which is the limiting position of the secant as the point M_1 tends to the point M_0 along the curve. This line is called tangent to the curve K at the point M_0.

Thus, if a function $y = f(x)$ is differentiable at a point x_0, then its graph has a tangent at the point $M_0(x_0, f(x_0))$ whose slope is equal to $f'(x_0)$.

The converse is also true: If the graph of the function $y = f(x)$, $x \in (a, b)$ has a tangent at a point $M_0(x_0, f(x_0))$, then the function f is differentiable at the point x_0, and its derivative is equal to the slope of the tangent at this point. The aforegoing makes it possible to give the following geometrical interpretation of the derivative: The derivative of a function $y = f(x)$ at a point x_0 is equal to the slope of the tangent to the graph of the function at the point $(x_0, f(x_0))$.

4. Computing the Derivative by Its Definition. Proceeding from the definition of the derivative, let us formulate the following rule for finding the derivative of a function at a given point.

In order to compute the derivative of a function $f(x)$ at a point x_0, it is necessary

(1) to find the difference $f(x) - f(x_0)$;

(2) to form the difference ratio $\frac{f(x)-f(x_0)}{x-x_0}$;

(3) to evaluate the limit of the difference ratio as $x \to x_0$:

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0).$$

Let us clarify this rule for finding the derivative by considering a few examples.

Example 1. Let $f(x) = C$, $x \in R$, where C is a constant. Find the derivative $f'(x)$.

△ (1) We find the difference $f(x) - f(x_0) = C - C = 0$.

(2) Form the difference ratio $\frac{f(x)-f(x_0)}{x-x_0} = \frac{0}{x-x_0} = 0$.

(3) Evaluate the limit

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} 0 = 0.$$

Thus, $f'(x_0) = (C)' = 0$ and, since x_0 is arbitrary, we write: $f'(x) = (C)' = 0$, that is, the derivative of a constant is equal to zero. ▲

Example 2. Find the derivative of the linear function $f(x) = kx + b$.

△ (1) We find the difference:

$$f(x) - f(x_0) = (kx + b) - (kx_0 + b) = k(x - x_0).$$

(2) Form the difference ratio

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{k(x-x_0)}{x-x_0}=k.$$

(3) The limit of this ratio is equal to k for any x_0. Thus $f'(x_0)=k$ or $f'(x)=(kx+b)'=k$. ▲

Example 3. Given a function $f(x)=x^3$, $x\in R$. Find $f'(x)$.

△ (1) We find the difference:

$$f(x)-f(x_0)=x^3-x_0^3=(x-x_0)(x^2+xx_0+x_0^2).$$

(2) Form the difference ratio

$$\frac{f(x)-f(x_0)}{x-x_0}=\frac{(x-x_0)(x^2+xx_0+x_0^2)}{x-x_0}=x^2+xx_0+x_0^2.$$

(3) Evaluate the limit

$$\lim_{x\to x_0}(x^2+xx_0+x_0^2)=3x_0^2.$$

Thus $f'(x_0)=3x_0^2$ or $f'(x)=3x^2$, $x\in R$. ▲

5. Continuity of a Differentiable Function. Let us formulate and prove the necessary condition for the existence of a derivative.

Theorem 6.1.1. *If a function $f(x)$ has a derivative at a point x_0, then it is continuous at this point.*

□ By hypothesis, the function $f(x)$ is differentiable at the point x_0, that is, there exists the limit

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}. \tag{1}$$

Using the property of the limit (see 5.6), equality (1) for x from a certain neighbourhood of the point x_0 can be written as follows:

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)+\alpha(x), \tag{2}$$

where $\lim\limits_{x\to 0}\alpha(x)=0$. Multiplying equality (2) by $(x-x_0)$, we find that a function differentiable at the point x_0 is representable in the neighbourhood of this point in the form

$$f(x)=f(x_0)+[f'(x_0)+\alpha(x)](x-x_0), \tag{3}$$

where $\lim\limits_{x\to x_0}\alpha(x)=0$. Passing to the limit in equality (3), as $x\to x_0$, we get

$$\lim_{x\to x_0}f(x)=f(x_0) \tag{4}$$

which just means that the function $f(x)$ is continuous at the point x_0. ■

Note. It follows from the above proved theorem that if a function is not continuous at some point, then it has no derivative at this point (proved by contradiction). Thus, the continuity of a function at a point is a necessary condition for its differentiability at this point. It should be further noted that the continuity of a function at a point is not a sufficient condition for the derivative of this function to exist at the point under consideration, that is, the continuity of a function at a point does not necessarily mean its differentiability at this point.

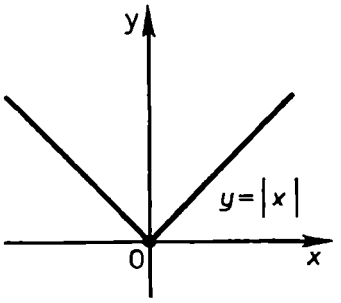

Fig. 100

Example. Find the derivative of the function $f(x) = |x|$.

△ Using the definition of the absolute value, we write the given function (see Fig. 100) in the form

$$f(x) = \begin{cases} x \text{ if } x > 0, \\ 0 \text{ if } x = 0, \\ -x \text{ if } x < 0. \end{cases}$$

Then, using the formula for the derivative of a linear function, we have:

$$f'(x) = (|x|)' = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ if } x < 0. \end{cases}$$

Making use of the definition of derivative, let us prove that the function $f(x) = |x|$ has no derivative at the point $x = 0$. Since $f(x) = -x$ for $x \leqslant 0$, we get:

$$\lim_{x \to -0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to -0} \frac{-x}{x} = -1.$$

Since $f(x) = x$ for $x \geqslant 0$, we have

$$\lim_{x \to +0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to +0} \frac{x}{x} = 1.$$

Thus, there is no limit of difference ratio at the point $x = 0$. And this means that the function $f(x) = |x|$ has no derivative at this point. Consequently, we have shown that the function $f(x) = |x|$, which is continuous for all $x \in R$, is nondifferentiable at the point $x = 0$. ▲

Exercises

1. A point performs rectilinear motion according to the law $x(t) = v_0 t + at^2/2$. Find the instantaneous velocity of this point: (a) for $t = 0$, (b) at time instant t_0.

2. A point performs rectilinear motion according to the law $x(t) = 2t^2 - 3t + 5$. Find the instantaneous velocity of this point: (a) for $t = 0$; (b) at time $t = 3c$.

3. Find the instantaneous velocity of a body at instant t_0 moving according to the law:

(a) $x(t) = 4\sqrt{t}$; (b) $x(t) = \frac{1}{5+t}$; (c) $x(t) = \frac{1}{3}t^3 + 2\sqrt{t}$.

4. Find the derivatives of the following functions at the points $x = x_0$, $x = 1$, and $x = 3$:

(a) $f(x) = x^2$; (b) $f(x) = 3x^2 + 4$; (c) $f(x) = (x+5)^2$;

(d) $f(x) = 2x^3 + 4x + 4$; (e) $f(x) = 9\sqrt{x}$; (f) $f(x) = 4 - x^3$;

(g) $f(x) = \frac{1}{2\sqrt{x}} - 3x^2$.

6.2. The Derivative of a Sum, Difference, Product, and a Quotient of Functions

1. The Derivative of a Sum and a Difference of Functions.

Theorem 6.2.1. *If two functions $u(x)$ and $v(x)$ are differentiable on some interval, then on this interval*

$$(u(x) \pm v(x))' = u'(x) \pm v'(x), \tag{1}$$

that is, the derivative of the sum (difference) of two functions is equal to the sum (difference) of the derivatives of these functions.

□ Let x_0 be a point belonging to the interval (a, b). Introducing the notation $f(x) = u(x) + v(x)$, by the definition of the derivative at the point x_0, we get:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(u(x) + v(x)) - (u(x_0) + v(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{u(x) - u(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{v(x) - v(x_0)}{x - x_0} = u'(x_0) + v'(x_0),$$

where we have taken advantage of the limit of a sum. Thus, $f'(x_0) = u'(x_0) + v'(x_0)$. By virtue of the arbitrarity of $x_0 \in (a, b)$, this relation can be written in the form

$$f'(x) = (u(x) + v(x))' = u'(x) + v'(x).$$

For the case of difference the theorem is proved in a similar way. Thereby relation (1) has been proved. ■

Analogously, we prove that the derivative of the sum (difference) of any finite number of functions is equal to the sum (difference) of the derivatives of these functions.

Example. Find the derivatives of the given functions:

(a) $(x^2+x-7)'=(x^2)'+(x)'-(7)'=2x+1$;

(b) $(x^3+\sqrt{x}-x+15)'=(x^3)'+(\sqrt{x})'-(x)'+(15)'$

$$=3x^2+\frac{1}{2\sqrt{x}}-1.$$

2. The Derivative of the Product of Two Functions.

Theorem 6.2.2. *If two functions $u(x)$ and $v(x)$ are differentiable on some interval, then on this interval*

$$(u(x)\,v(x))'=u'(x)\,v(x)+u(x)\,v'(x) \tag{1}$$

or in a brief notation

$$(uv)'=u'v+uv'.$$

□ Let x_0 be a point belonging to the interval (a, b). Introducing the notation $f(x)=u(x)\,v(x)$, by the definition of the derivative at the point x_0, we have:

$$\begin{aligned}f'(x_0)&=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{u(x)\,v(x)-u(x_0)\,v(x_0)}{x-x_0}\\&=\lim_{x\to x_0}\frac{(u(x)-u(x_0))\,v(x)+u(x_0)\,(v(x)-v(x_0))}{x-x_0}\\&=v(x_0)\lim_{x\to x_0}\frac{u(x)-u(x_0)}{x-x_0}+u(x_0)\lim_{x\to x_0}\frac{v(x)-v(x_0)}{x-x_0}\\&=v(x_0)\,u'(x_0)+u(x_0)\,v'(x_0),\end{aligned}$$

where we have made use of the properties of the limit of a sum and a product. Thus, $f'(x_0)=u'(x_0)\,v(x_0)+u(x_0)\,v'(x_0)$. Since $x_0\in(a, b)$ is arbitrary, we may write this relation in the form $f'(x)=u'(x)\,v(x)+u(x)\,v'(x)$. Thus, relation (1) has been proved. ■

Corollary. A constant factor can be taken outside the sign of a derivative, that is, $(cu)'=cu'$, where $u=u(x)$ is a differentiable function.

The validity of the assertion follows from the rule for differentiating the product of functions and the equality $(c)'=0$.

Example. Find the derivatives of the given functions:

(a) $\left(\frac{x^3}{7}\right)'=\frac{1}{7}(x^3)'=\frac{3x^2}{7}$;

(b) $(\sqrt[7]{x^2})'=(x^3\sqrt{x})'=(x^3)'\sqrt{x}+x^3(\sqrt{x})'$

$$=3x^2\sqrt{x}+\frac{x^3}{2\sqrt{x}}=\frac{7}{2}x^{\frac{5}{2}}.$$

3. The Derivative of the Quotient of Two Functions.

Theorem 6.2.3. *Let two functions $u(x)$ and $v(x)$ be differentiable on the interval (a, b). If $v(x) \neq 0$ for any $x \in (a, b)$, then*

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v^2(x)}. \tag{1}$$

□ Let x_0 be a point belonging to the interval (a, b). Introducing the notation $f(x) = \frac{u(x)}{v(x)}$, by the definition of the derivative at the point x_0, we have:

$$\begin{aligned}
f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{u(x)\,v(x_0) - v(x)\,u(x_0)}{v(x)\,v(x_0)\,(x - x_0)} \\
&= \frac{1}{v^2(x_0)} \cdot \lim_{x \to x_0} \frac{u(x)\,v(x_0) - u(x_0)\,v(x)}{x - x_0} \\
&= \frac{1}{v^2(x_0)} \lim_{x \to x_0} \frac{(u(x) - u(x_0))\,v(x_0) - u(x_0)\,(v(x) - v(x_0))}{x - x_0} \\
&= \frac{1}{v^2(x_0)} \lim_{x \to x_0} \left(\frac{u(x) - u(x_0)}{x - x_0} v(x_0) - u(x_0) \frac{v(x) - v(x_0)}{x - x_0} \right) \\
&= \frac{u'(x_0)\,v(x_0) - u(x_0)\,v'(x_0)}{v^2(x_0)},
\end{aligned}$$

where we have made use of the property of the limit of products and a sum. Since $x_0 \in (a, b)$ is arbitrary, the obtained relation can be written in the form

$$f'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v^2(x)},$$

and thus relation (1) has been proved. ■

Example. Find the derivatives of the given functions:

(a) $\left(\frac{x-7}{1+x}\right)' = \frac{(x-7)'(1+x) - (x-7)(1+x)'}{(1+x)^2} = \frac{1+x-(x-7)}{(1+x)^2} = \frac{8}{(1+x)^2}$;

(b) $\left(\frac{x^2}{1-x}\right)' = \frac{(x^2)'(1-x) - x^2(1-x)'}{(1-x)^2} = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{x(2-x)}{(1-x)^2}$.

Exercises

1. Find the derivative for each of the given functions:

(a) $f(x) = 2x + 8$; (b) $g(x) = 3x^2 + x + 7$;

(c) $h(x) = 4\sqrt{x} + x^2 - 5$; (d) $v(x) = x^4 + \sqrt{x}$;

(e) $y(x) = x^4 + x^2 + \sqrt{x} + 9$; (f) $u(x) = 3 - 7x + 8x^3$;

(g) $w(x) = 8x - 9 + x^4 - 2x^3$.

2. Find the derivative for each of the given functions:

(a) $f(x)=(x-3)(x+5)$; (b) $f(x)=x^3(x-\sqrt{x})$;

(c) $f(x)=2x\left(x^2-\frac{3}{5}\sqrt{x}\right)$; (d) $f(x)=(x^3-2x+1)(1-5x-8x^2)$;

(e) $u(t)=(t^3-3\sqrt{t})(1-t)$.

3. Prove that $(uvw)'=u'vw+uv'w+uvw'$.

4. Find the derivatives (at the point $x=1$) of the following functions:

(a) $f(x)=\frac{3x-1}{5x+7}$; (b) $g(x)=\frac{x^2-1}{3+5x}$;

(c) $h(x)=\frac{x^3}{x^2-4x}$; (d) $v(x)=\frac{\sqrt{x}}{2x-x^3}$;

(e) $u(x)=\frac{x^2+\sqrt{x}}{x^2-5}$; (f) $w(x)=\frac{\sqrt{x}-3x^2+5x^3}{x-4x^3}$.

6.3. The Derivative of a Composite, Inverse, and Parametrically Represented Functions

1. The Derivative of a Composite Function.

Theorem 6.3.1. *Let the function $y=g(x)$, $x\in(a, b)$, have a derivative at the point $x_0\in(a, b)$, and the function $z=\varphi(y)$ have a derivative at the point $y_0=g(x_0)$. Then the composite function $z(x)=\varphi(g(x))$ has a derivative at the point x_0 which is computed by the formula*

$$z'(x_0)=\varphi'(y_0)\,g'(x_0) \text{ or } \frac{dz}{dx}=\frac{dz}{dy}\cdot\frac{dy}{dx}.$$

□ The composite function $z(x)=\varphi(g(x))$ has a derivative at the point x_0 if there exists the limit

$$\lim_{x\to x_0}\frac{\varphi(g(x))-\varphi(g(x_0))}{x-x_0}=z'(x_0). \tag{1}$$

By the definition of the limit of a function at a point, limit (1) exists if for any sequence $\{x_n\}$ such that $x_n\to x_0$, $x_n\neq x_0$, the sequence

$$a_n=\frac{\varphi(g(x_n))-\varphi(g(x_0))}{x_n-x_0},\quad n\in N, \tag{2}$$

has a limit. Limit (1) is equal to the limit of sequence (2). Let us represent it in the following form:

$$a_n=\frac{\varphi(g(x_n))-\varphi(g(x_0))}{g(x_n)-g(x_0)}\cdot\frac{g(x_n)-g(x_0)}{x_n-x_0}=\frac{\varphi(y_n)-\varphi(y_0)}{y_n-y_0}\cdot\frac{g(x_n)-g(x_0)}{x_n-x_0},$$

where $y_n=g(x_n)$ is the sequence of values of the function $y=g(x)$. Since the function $z=\varphi(y)$ has a derivative at the point y_0,

and the function $y = g(x)$ has a derivative at the point x_0, there exist the limits

$$\lim_{n\to\infty} \frac{\varphi(y_n)-\varphi(y_0)}{y_n-y_0} = \varphi'(y_0) \text{ and } \lim_{n\to\infty} \frac{g(x_n)-g(x_0)}{x_n-x_0} = g'(x_0).$$

Then, by the theorem on the limit of the product of sequences, we find that the limit of sequence (2) exists and is computed by the formula

$$\lim_{n\to\infty} \frac{\varphi(g(x_n))-\varphi(g(x_0))}{x_n-x_0} = \lim_{n\to\infty} \left[\frac{\varphi(y_n)-\varphi(y_0)}{y_n-y_0} \cdot \frac{g(x_n)-g(x_0)}{x_n-x_0}\right]$$

$$= \lim_{n\to\infty} \frac{\varphi(y_n)-\varphi(y_0)}{y_n-y_0} \cdot \lim_{n\to\infty} \frac{g(x_n)-g(x_0)}{x_n-x_0} = \varphi'(y_0)\, g'(x_0).$$

Thus,

$$z'(x_0) = \varphi'(y_0)\, g'(x_0)$$

or in other notation

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}. \quad \blacksquare \tag{3}$$

Note. The above proof does not hold if there exists a sequence $\{x_n\}$ converging to x_0 for which $g(x_n) - g(x_0) = 0$, $n \in N$. However, in this case

$$g'(x_0) = \lim_{n\to\infty} \frac{g(x_n)-g(x_0)}{x_n-x_0} = \lim_{n\to\infty} \frac{0}{x_n-x_0} = 0$$

and

$$z'(x_0) = \lim_{n\to\infty} \frac{\varphi(g(x_n))-\varphi(g(x_0))}{x_n-x_0} = \lim_{n\to\infty} \frac{\varphi(g(x_0))-\varphi(g(x_0))}{x_n-x_0} = 0,$$

so that formula (3) is true as before.

Example 1. Find the derivative of the function $f(x) = (x^4 + 5x^2 + 9)^2$.

△ We shall regard the given function as a composite one, namely, as the superposition of the functions $z = y^2$ and $y = x^4 + 5x^2 + 9$. Then, according to formula (3), we get:

$$f'(x) = \frac{dz}{dy} \cdot \frac{dy}{dx} = (y^2)'(x^4+5x^2+9)' = 2y(4x^3+10x)$$

$$= 2(x^4+5x^2+9)(4x^3+10x) = 4x(x^4+5x^2+9)(2x^2+5). \ \blacktriangle$$

Example 2. Find the derivative of the function $h(x) = \left(\frac{2x}{3x-1}\right)^3$.

△ We represent the given function as a superposition of two functions $z = y^3$ and $y = \frac{2x}{3x-1}$. According to formula (3), we get:

$$h'(x) = \frac{dz}{dy} \cdot \frac{dy}{dx} = (y^3)' \left(\frac{2x}{3x-1}\right)' = 3y^2 \cdot \frac{(2x)'(3x-1)-2x(3x-1)'}{(3x-1)^2}$$

$$= 3\left(\frac{2x}{3x-1}\right)^2 \cdot \left(-\frac{2}{(3x-1)^2}\right) = -\frac{24x^2}{(3x-1)^3}. \ \blacktriangle$$

2. The Derivative of an Inverse Function.

Theorem 6.3.2. *If a function* $y = f(x)$, $x \in (a, b)$, *and its inverse* $x = f^{-1}(y)$ *have derivatives, and* $\frac{df}{dx} \neq 0$, *then*

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}} \text{ or } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ or } x' = \frac{1}{y'}. \quad (1)$$

□ Consider the composite function $f^{-1}(f(x))$, $x \in (a, b)$. By the definition of an inverse function,

$$f^{-1}(f(x)) = x$$

for any $x \in (a, b)$. Making use of the theorem of the derivative of a composite function, we get

$$\frac{df^{-1}(y)}{dy} \cdot \frac{df(x)}{dx} = 1,$$

whence there follows the validity of formula (1). ■

Example. Find the derivative of the function $y = x^{1/3}$, $x > 0$.

△ The given function is inverse to the function $x = y^3$, $y > 0$. Consequently,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2} = \frac{1}{3x^{2/3}} = \frac{1}{3} x^{-2/3}. \text{ ▲}$$

3. The Derivative of a Function Represented Parametrically. Let the functions $x = x(t)$ and $y = y(t)$ be defined in some neighbourhood of the point t_0, and one of them, say $x = x(t)$, has the inverse $t = t(x)$ in this neighbourhood. Then the composite function $y(t(x))$ is said to be a function represented parametrically by the formulas $x = x(t)$ and $y = y(t)$. For example, the functions $x = R \cos t$ and $y = R \sin t$, $0 \leqslant t \leqslant \pi$, where $R > 0$, are a parametric representation of the function $y = R\sqrt{1 - x^2}$. Its graph is one-half of a circle located in the upper half-plane (Fig. 101).

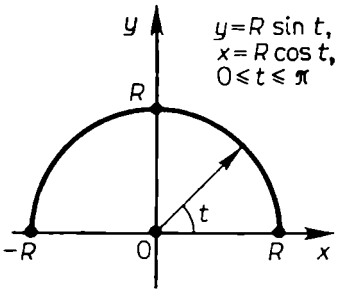

Fig. 101

Let the functions $x = x(t)$ and $y = y(t)$ be differentiable, and $x'(t) \neq 0$. We find the derivative $\frac{df}{dx}$. According to the rule for differentiating composite functions, we have:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Since

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}},$$

we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{1}$$

Sometimes, formula (1) is written in the form

$$\frac{dy}{dx} = \frac{y'_t}{x'_t}.$$

Example. Find the derivative of the parametrically represented function:

$$x = 2t^3 + 3t^2 + 1, \quad y = \sqrt{t}.$$

△ We make use of formula (1) for finding the derivative:

$$\frac{dy}{dx} = \frac{y'_t}{x'_t} = \frac{\frac{1}{2\sqrt{t}}}{6t^2+6t} = \frac{1}{12t\sqrt{t}(t+1)}. \ ▲$$

Exercises

1. Find the derivative of each of the composite functions:

(a) $y = (21 - 15x^2 + x^3)^2$; (b) $y = (x^3 - 7)^4$, (c) $y = \left(\frac{1+x^2}{x^2+x}\right)^3$;

(d) $y = \left(\frac{x^2+x+1}{x^2-3x-3}\right)^2$.

2. Find the derivatives $\frac{dy}{dx}$ of the following parametrically represented functions:

(a) $x = 1 - t^2$, $y = t - t^3$; (b) $x = \frac{t+1}{t}$, $y = \frac{t-1}{t}$;

(c) $x = \sqrt{t}$, $y = t^2$; (d) $x = \frac{1+t^3}{t^2-1}$, $y = \frac{1}{t^2-1}$.

6.4. Derivatives of Certain Elementary Functions

1. The Derivative of an Exponential Function. Let us consider the function $f(x) = e^x$, $x \in R$.

Theorem 6.4.1. *The function e^x has a derivative at every point of the number line, and its derivative is computed by the formula*

$$(e^x)' = e^x. \tag{1}$$

□ Let x_0 be a point on the number line. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0}$$

$$= e^{x_0} \cdot \lim_{x \to x_0} \frac{e^{x - x_0} - 1}{x - x_0} = e^{x_0} \cdot \lim_{t \to 0} \frac{e^t - 1}{t}, \qquad (2)$$

where $x - x_0$ is denoted by t. In 5.8 it was proved that

$$\lim_{t \to 0} \frac{e^t - 1}{t} = 1. \qquad (3)$$

Using equality (3), we find from formula (2) that $f'(x_0) = e^{x_0}$ and, since x_0 is arbitrary, we may write:

$$f'(x) = (e^x)' = e^x. \blacksquare$$

Let us now consider the exponential function a^x, $x \in R$, with an arbitrary base $a > 0$, $a \neq 1$.

Since $a^x = e^{x \ln a}$, by the formula for the derivative of a composite function, we find:

$$f'(x) = (e^{x \ln a})' = e^{x \ln a} (x \ln a)' = e^{x \ln a} \ln a = a^x \ln a.$$

Thus,

$$(a^x)' = a^x \ln a. \qquad (4)$$

Example 1. Find the derivative of the function $y = e^{x^3 + 4x}$.

△ Making use of the formula for the derivative of a composite function and formula (1), we find the derivative of the given function:

$$y' = (e^{x^3 + 4x}) = (e^z)' (x^3 + 4x)' = e^z (3x^2 + 4) = (3x^2 + 4) e^{x^3 + 4x}. \blacktriangle$$

Example 2. Differentiate the function $y = 8^{3x^4 - 4x + 10}$.

△ Using formula (4) and the formula for the derivative of a composite function, we find

$$y' = (8^z)' (3x^4 - 4x + 10)' = 8^z \ln 8 (12x^3 - 4)$$

$$= 8^{(3x^4 - 4x + 10)} \cdot \ln 8 (12x^3 - 4). \blacktriangle$$

2. The Derivative of a Logarithmic Function. The function $y = \log_a x$, $x \in R_+$ is the converse of the function $x = a^y$. Therefore, by the formula for the derivative of an inverse function, we have:

$$(\log_a x)' = \frac{1}{(a^y)'} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

Consequently,

$$(\log_a x)' = \frac{1}{x \ln a}. \qquad (1)$$

Note separately the important particular case $a = e$:

$$(\ln x)' = \frac{1}{x}. \tag{2}$$

Example 1. Differentiate the function

$$y = \ln (x^3 + 3x^2 + 6).$$

△ Using the formula for the derivative of a composite function and formula (2), we get:

$$y' = (\ln (x^3 + 3x^2 + 6))' = (\ln z)' (x^3 + 3x^2 + 6)' = \frac{3x^2 + 6x}{z}$$

$$= \frac{3x^2 + 6x}{x^3 + 3x^2 + 6}. \blacktriangle$$

Example 2. Find the derivative of the function

$$y = (x^4 + 9) \log_2 (x + 1).$$

△ Using the formulas for the derivative of a composite function and a product, and also formula (1), we find:

$$y' = (x^4 + 9)' \log_2 (x + 1) + (x^4 + 9) (\log_2 (x + 1))'$$
$$= 4x^3 \log_2 (x + 1) + (x^4 + 9) (\log_2 z)' (x + 1)'$$
$$= 4x^3 \log_2 (x + 1) + \frac{x^4 + 9}{(x + 1) \ln 2}. \blacktriangle$$

3. The Derivative of a Power Function. In the case of an arbitrary $\alpha \in R$, the power function $y = x^\alpha$ is considered for $x > 0$. In this case

$$x^\alpha = e^{\alpha \ln x}.$$

By the rule for differentiating composite functions, we find:

$$(x^\alpha)' = (e^{\alpha \ln x})' = (e^z)' (\alpha \ln x)' = e^z \frac{\alpha}{x} = \frac{\alpha e^{\alpha \ln x}}{x} = \alpha x^{\alpha - 1}.$$

Consequently

$$(x^\alpha)' = \alpha x^{\alpha - 1}. \tag{1}$$

This formula also remains valid for the case when a power function is defined throughout the number line (for instance, for natural α).

Example. Find the derivative of each of the given functions:

(a) Let $y = x$, then $y' = (x)' = 1 \cdot x^{1-1} = 1$;

(b) let $y = x^{20}$, then $y' = (x^{20})' = 20x^{20-1} = 20x^{19}$;

(c) if $y = \sqrt[5]{x}$, then $y' = (\sqrt[5]{x})' = (x^{1/5})' = \frac{1}{5} x^{1/5-1} = \frac{1}{5} x^{-\frac{4}{5}}$;

(d) if $y = x^{\sqrt{7}}$, then $y' = (x^{\sqrt{7}})' = \sqrt{7}\, x^{\sqrt{7}-1}$;

(e) if $y = \sqrt[9]{x^5}$, then $y' = (\sqrt[9]{x^5})' = (x^{5/9})' = \frac{5}{9} x^{\frac{5}{9}-1} = \frac{5}{9} x^{-\frac{4}{9}}$;

(f) let $y = x^8 + 3x^{15} + 20x^{19}$, then $y' = 8x^7 + 45x^{14} + 380x^{18}$;

(g) if $y = 19x^6 + \sqrt[3]{x} - \frac{9}{x^{30}}$, then $y' = 114x^5 + \frac{1}{3} x^{-\frac{2}{3}} + 270x^{-31}$.

4. The Derivatives of Trigonometric Functions. Consider the function $f(x) = \sin x$, $x \in R$.

Theorem 6.4.2. *The sine has a derivative at every point of the number line, and its derivative is computed by the formula*

$$(\sin x)' = \cos x. \tag{1}$$

□ Let x_0 be a point on the number line. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} = \lim_{x \to x_0} \frac{2 \cos \frac{x + x_0}{2} \sin \frac{x - x_0}{2}}{x - x_0}$$

$$= \cos x_0 \cdot \lim_{x \to x_0} \frac{\sin \left(\frac{x - x_0}{2} \right)}{\left(\frac{x - x_0}{2} \right)} = \cos x_0 \cdot \lim_{t \to 0} \frac{\sin t}{t},$$

where we have used the property of the limit of a product and replaced the variable x by t by the formula $t = \frac{x - x_0}{2}$. Since in 5.8 it was proved that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1,$$

we have $f'(x_0) = \cos x_0$. Since x_0 is arbitrary, we establish finally: $f'(x) = (\sin x)' = \cos x$. ■

Since $\cos x = \sin \left(\frac{\pi}{2} - x \right)$, by the formula for the derivative of a composite function, we find:

$$(\cos x)' = \left(\sin \left(\frac{\pi}{2} - x \right) \right)' = (\sin z)' \left(\frac{\pi}{2} - x \right)' = \cos z \cdot (-1)$$

$$= -\cos \left(\frac{\pi}{2} - x \right) = -\sin x.$$

Thus

$$(\cos x)' = -\sin x. \tag{2}$$

Since $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$, using formula (1) and (2) and the rule for differentiating a quotient, we find:

$$(\tan x)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x},$$

$$(\cot x)' = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Example. Compute the derivatives of the following functions:

(a) $(\sin 4x)' = \cos 4x \cdot (4x)' = 4 \cos 4x$;

(b) $(\sin 8x^2)' = \cos 8x^2 \cdot (8x^2)' = 16x \cos 8x^2$;

(c) $(\sin^5 3x)' = 5 \sin^4 3x \cdot (\sin 3x)' = 5 \sin^4 3x \cos 3x \, (3x)' = 15 \sin^4 3x \cos 3x$;

(d) $(\cos 7x)' = -\sin 7x \cdot (7x)' = -7 \sin 7x$;

(e) $(\cos^2 5x)' = 2 \cos 5x \, (\cos 5x)' = 2 \cos 5x \, (-\sin 5x) \, (5x)' = -5 \sin 10x$;

(f) $(\tan 2x)' = \frac{1}{\cos^2 2x} (2x)' = \frac{2}{\cos^2 2x}$;

(g) $(\tan^2 (4x-1))' = 2 \tan (4x-1) [\tan (4x-1)]' = 2 \tan (4x-1) \times \frac{1}{\cos^2 (4x-1)} (4x-1)' = \frac{8 \tan (4x-1)}{\cot^2 (4x-1)} = \frac{8 \sin (4x-1)}{\cos^3 (4x-1)}$;

(h) $(\cot^7 x^2)' = 7 \cot^6 x^2 (\cot x^2)' = 7 \cot^6 x^2 \left(-\frac{1}{\sin^2 x^2}\right) (x^2)' = \frac{-14x \cos^6 x^2}{\sin^2 x^2}$.

5. The Derivatives of Inverse Trigonometric Functions. Consider the function $y = \arcsin x$, $x \in [-1, 1]$.

Theorem 6.4.3. *The arcsine is a function differentiable on the interval* $(-1, 1)$, *and its derivative is computed by the formula*

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}. \tag{1}$$

□ Since the function $y = \arcsin x$, $x \in [-1, 1]$ is the converse of the function $x = \sin y$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by the rule for differentiating inverse functions, we have

$$(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}},$$

where the radical is taken with the plus sign since cos y is positive on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus,

$$(\arcsin x')=\frac{1}{\sqrt{1-x^2}}. \blacksquare$$

The functions arcsin x and arccos x are related by the following expression (see 5.3, subsection 1)

$$\arcsin x=\arccos x=\frac{\pi}{2},$$

that is,

$$\arccos x=\frac{\pi}{2}-\arcsin x. \tag{2}$$

Differentiating (2), we find:

$$(\arccos x)'=-(\arcsin x)'=-\frac{1}{\sqrt{1-x^2}}.$$

Thus, we have proved that the arccosine has a derivative on the interval $(-1, 1)$ which is computed by the formula

$$(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}. \tag{3}$$

Since the function $y=\arctan x$, $x\in R$ is the converse of the function $x=\tan y$, $y\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have (by the rule for differentiating inverse functions)

$$(\arctan x)'=\frac{1}{(\tan y)'}=\cos^2 y=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Thus,

$$(\arctan x)'\frac{1}{1+x^2}.$$

Thus, we have found that the arc tangent is a function differentiable everywhere on the number line, and its derivative is found by the formula

$$(\arctan x)'=\frac{1}{1+x^2}. \tag{4}$$

The functions arctan x and arccot x are related as follows (see 5.3., subsection 1)

$$\arctan x+\operatorname{arccot} x=\frac{\pi}{2}.$$

That is,

$$\operatorname{arccot} x=\frac{\pi}{2}-\arctan x. \tag{5}$$

Differentiating (5), we find:

$$(\operatorname{arccot} x)' = -(\arctan x)' = -\frac{1}{1+x^2}.$$

Consequently, the arc cotangent is a function differentiable throughout the number line, and its derivative is found by the formula

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}. \tag{6}$$

Example. Compute the derivative of each of the given functions:

(a) $(\arcsin \sqrt{x})' = (\arcsin z)'(\sqrt{x})' = \frac{1}{\sqrt{1-z^2}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$;

(b) $(x \arcsin x)' = (x)' \arcsin x + x (\arcsin x)' = \arcsin x + \frac{x}{\sqrt{1-x^2}}$;

(c) $((\arccos x)^2)' = (z^2)' (\arccos x)' = 2z\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{2 \arccos x}{\sqrt{1-x^2}}$;

(d) $((\arctan \sqrt{x})^4)' = (z^4)' (\operatorname{arccot} v)' (\sqrt{x})' = 4z^3 \cdot \frac{1}{1+v^2} \cdot \frac{1}{2\sqrt{x}} = \frac{2(\arctan \sqrt{x})^3}{(1+x)\sqrt{x}}$;

(e) $\left(\operatorname{arccot}\left(\frac{x^2}{1+x}\right)\right)' = (\operatorname{arccot} z)' \left(\frac{x^2}{1+x}\right)' = \frac{-1}{1+z^2} \cdot \frac{2x(1+x)-x^2}{(1+x)^2} = \frac{-1}{1+\left(\frac{x^2}{1+x}\right)^2} \cdot \frac{x^2+2x}{(1+x)^2} = \frac{-(x^2+2x)}{(1+x)^2+x^4} = \frac{-x(x+2)}{x^4+x^2+2x+1}$.

For obviousness, we tabulate the derivatives of simplest elementary functions (see Table 1).

Table 1

1. $(C)' = 0$	7. $(\tan x)' = \frac{1}{\cos^2 x}$
2. $(x^\alpha)' = \alpha x^{-1}$	8. $(\cot x)' = -\frac{1}{\sin^2 x}$
3. $(a^x)' = a^x \ln a$	9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
4. $(\log_a x)' = \frac{1}{x \ln a}$	10. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
5. $(\sin x)' = \cos x$	11. $(\arctan x)' = \frac{1}{1+x^2}$
6. $(\cos x)' = -\sin x$	12. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$

Exercises

1. Find the derivative of each of the following functions:

(a) $y=(x+4)\,e^{x}$; (b) $y=x^{5}e^{x^{7}+3x^{2}}$; (c) $y=\dfrac{x^{4}+1}{e^{x}}$;

(d) $y=x^{2}\cdot 2^{8x+3x^{2}}$; (e) $y=(x^{2}+1)\,e^{\frac{3}{x}}$.

2. Find the derivative of the given functions:

(a) $y=(x^{2}+7)\ln x^{6}$; (b) $y=\dfrac{\ln x}{x^{2}-1}$; (c) $y=\dfrac{1+x^{3}}{8\ln x}$;

(d) $y=a^{x^{7}}\log(x^{2}+4x+12)$.

3. Find the derivative of each of the given functions:

(a) $y=x^{200}$; (b) $y=-x^{8}$; (c) $y=\sqrt[15]{x^{7}}$;

(d) $y=\sqrt[6]{x+\sqrt[3]{x}}$; (e) $y=\dfrac{1}{6\cdot\sqrt[3]{4x^{2}}}$; (f) $y=\dfrac{x^{3}+3x+x^{7}}{3\sqrt[3]{x^{2}+1}}$.

4. Find the derivative of each of the given functions:

(a) $y=\sin 2x$; (b) $y=\sin ax$;
(c) $y=\cos 3x$; (d) $y=\cos ax$;
(e) $y=\sin 2x-\cos 3x$; (f) $y=x\cos x$;
(g) $y=x\sin x$; (h) $y=\sin x\cos x$;
(i) $y=\sin 2x\cos 3x$; (j) $y=\sin ax\cos bx$;
(k) $y=\sin^{2}x$; (l) $y=\tan 2x-\cot 2x$;
(m) $y=x\cot x$; (n) $y=\tan^{3}x+\tan 3x$;
(o) $y=\cot 2x-\tan^{2}2x$; (p) $y=(\sin x+\cos x)^{2}$;
(q) $y=\sin^{2}x-\cos^{2}x$.

5. Find the derivative of each of the given functions:

(a) $y=\arcsin 7x$; (b) $y=\arcsin ax$; (c) $y=m\cdot\arcsin x$;
(d) $y=\arccos 4x$; (e) $y=\arccos ax$; (f) $y=m\cdot\arccos nx$;
(g) $y=\arctan 3x$; (h) $y=\arctan mx$; (i) $y=m\cdot\operatorname{arccot} nx$;
(j) $y=\sin^{n}ax$; (k) $y=\cos^{3}(x+\pi)$; (l) $y=\sin x+\cos 5x+\tan 7x$;

(m) $y=\dfrac{\tan 7x}{7}$; (n) $y=2\tan^{3}4x$; (o) $y=4\cot^{3}2x$;

(p) $y=\arcsin x^{3}$; (q) $y=x^{4}\arcsin x$.

6. Given: $y=\operatorname{arccot} x^{-1}$. Find y'.

7. Given: $y=\arcsin 2x+\arctan 3x+\arccos 2x+\operatorname{arccot} 3x$. Find y'.

8. Find the derivatives of the following functions:

(a) $f(x) = (x - 3x^2 - x^{11})(\sqrt{x} + 4x^7 - 9)$;

(b) $f(x) = (x^{13} + 7x^{11} + \sqrt[5]{x^2}) \ln x$;

(c) $f(t) = (t + \sqrt[3]{t})\, e^{t^2 - 4}$;

(d) $f(t) = (\ln t + \sqrt[3]{t^2})\, e^{47t^5}$.

6.5. Higher-order Derivatives

1. The Second Derivative. If a function $y = f(x)$ is differentiable on the interval (a, b), then every $x \in (a, b)$ is associated with $f'(x)$, that is, a new function $g(x) = f'(x)$ is defined on (a, b). If the function $g(x)$ is differentiable at a point $x_0 \in (a, b)$, then the derivative $g'(x_0)$ is called the second derivative (or the derivative of the second order) of the function $f(x)$ at the point x_0 and is denoted by $f''(x_0)$ or $\frac{d^2 f(x_0)}{dx^2}$. If the function $f(x)$ has a second derivative at any point of the interval (a, b), then, by definition

$$f''(x) = (f'(x))'.$$

Example 1. Find the second derivatives of the given functions:

(a) if $f(x) = 3x^2 + 2x - 3$, then $f'(x) = 6x + 2$, $f''(x) = (6x + 2)' = 6$;

(b) if $f(x) = x^2 \ln x$, then $f'(x) = (x^2)' \ln x + x^2 (\ln x)' = 2x \ln x + x^{(n)}$, $f''(x) = (2x \ln x + x)' = 2 \ln x + 2x (\ln x)' + 1 = 2 \ln x + 3$.

Similarly, we introduce the function $(f''(x))'$ which is called the third derivative (or the derivative of the third order) and is denoted by $f'''(x)$, and so on. In general, the nth derivative (or the derivative of the nth order) of the function $f(x)$ on the interval (a, b) is defined as the derivative of the derivative of the $(n-1)$st order. It is denoted by $f^{(n)}(x)$:

$$f^{(n)}(x) = (f^{(n-1)}(x))'.$$

The order of the derivative is written in parentheses in order not to confuse it with an exponent.

Example 2. Given a function $y = e^{kx}$ (k constant). Find its derivative of the nth order.

△ We compute in succession: $y' = ke^{kx}$, $y'' = k^2 e^{kx}$, $y''' = k^3 e^{kx}$, ... Hence $y^{(n)} = k^n e^{kx}$. ▲

Example 3. Given a function: $y = 3x^3 + 2x - 7$. Find its derivative of the fifth order.

△ We compute in succession: $y' = 9x^2 + 2$, $y'' = 18x$, $y''' = 18$, $y^{(4)} = 0$, $y^{(5)} = 0$. It is clear that all consecutive derivatives are also equal to zero.

2. Physical Meaning of the Second Derivative. Let us return to the problem on nonuniform motion of a material point along a straight line considered in 6.1. The law of motion is given by the function

$$x = f(t), \tag{1}$$

where x is the coordinate of the material point, and t, time. The instantaneous velocity of motion $v(t)$ is computed by the formula

$$v(t) = f'(t). \tag{2}$$

The velocity characterizes the rate of change in the coordinate of the material point with time. To describe the rate of change of the velocity itself, we introduce the acceleration $a(t)$ by the formula

$$a(t) = v'(t). \tag{3}$$

According to the definition of the derivative of the second order, from formulas (1)-(3) we find:

$$a(t) = f''(t) = x'', \tag{4}$$

that is, the second derivative of the coordinate is acceleration. This is the mechanical meaning of the second derivative. In physics, the role of second derivatives of the coordinates of material bodies is extremely important since they are directly related with the forces acting on bodies which is reflected by the principal dynamical law of motion—Newton's second law:

$$m \cdot a(t) = m \cdot x''(t) = F(t), \tag{5}$$

where m is the mass of a material point (particle) (a certain constant), $F(t)$ the force which is in the general case a function of time. From Newton's equation (5) it is seen that in a simple particular case, when a body is acted upon by a constant force, the body performs motion with a constant acceleration. Such a motion is called uniformly accelerated and is studied in detail in the course of school physics.

Exercises

1. Find the fourth derivative of the following functions:

(a) $y = \sin 7x$; (b) $y = (x + 8)^5$; (c) $y = \cos 3x$;

(d) $y = e^{2x} + 4x^3$; (e) $y = 3 + 2x^5 + 4e^x$; (f) $y = e^{4x} + \sin 9x$.

2. Find the second derivative of each of the following functions:

(a) $y = \tan 6x$; (b) $y = 2 \cot 3x$; (c) $y = \frac{1}{4} \ln 5x$;

(d) $y = a^{4x^2-7}$; (e) $y = \arcsin 8x$; (f) $y = \arctan (5x + 2)$.

3. How many times do we have to differentiate the function $y = (x^2 + 1)^{50}$ to get a polynomial of the thirtieth degree?

4. Prove that for the function $y = x^2 + e^x$ the equality $y^{(4)} = y^{(5)}$ holds.

6.6. L'Hospital's Rule

1. Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. If two functions $u(x)$ and $v(x)$, differentiable on the interval (a, b), vanish simultaneously at a point $c \in (a, b)$, then, when evaluating the limit $\lim\limits_{x \to c} \frac{u(x)}{v(x)}$ we may not use the theorem on the limit of a quotient since in this case there appears an indeterminate form $\frac{0}{0}$. In this case, we apply the following theorem which is given without proof:

Theorem 6.6.1. *Let two functions $u(x)$ and $v(x)$ be differentiable on the interval (a, b) and vanish at a point $x = c$. Then*

$$\lim_{x \to c} \frac{u(x)}{v(x)} = \lim_{x \to c} \frac{u'(x)}{v'(x)}, \tag{1}$$

provided that the last limit exists. Evaluating limits by formula (1) is called l'Hospital's rule for evaluating indeterminate forms $\frac{0}{0}$.

Similarly, for the case

$$\lim_{x \to \infty} u(x) = 0 \text{ and } \lim_{x \to \infty} v(x) = 0$$

the following equality is valid:

$$\lim_{x \to \infty} \frac{u(x)}{v(x)} = \lim_{x \to \infty} \frac{u'(x)}{v'(x)}, \tag{2}$$

provided that the latter limit exists. This formula is called l'Hospital's rule for evaluating indeterminate forms $\frac{0}{0}$ as $x \to \infty$. In formula (2), instead of the sign ∞, we may use the sign $+\infty$ or $-\infty$.

Example 1. Evaluate the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

△ For evaluating the given limit, it is impossible to apply the theorem on the limit of a quotient since in this case there appears an indeterminate form $\frac{0}{0}$. Indeed, $\lim\limits_{x \to 1} \ln x = \ln 1 = 0$ and

$\lim\limits_{x\to 1}(x-1)=0$. We evaluate the indeterminate form $\frac{0}{0}$ using l'Hospital's rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{x\to 1}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to 1}\frac{\left(\frac{1}{x}\right)}{1}=1. \blacktriangle$$

Example 2. Evaluate the limit

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}.$$

△ Since $\lim\limits_{x\to 0}(e^x-e^{-x})=1-1=0$ and $\lim\limits_{x\to 0}x=0$, there appears an indeterminate form $\frac{0}{0}$. We evaluate this indeterminate form by applying l'Hospital's rule:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{1}=2. \blacktriangle$$

Example 3. Evaluate the limit

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)}.$$

△ Since $\lim\limits_{x\to\infty}\sin\frac{1}{x}=\sin\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)=\sin 0=0$ and $\lim\limits_{x\to\infty}\frac{1}{x}=0$, there appears an indeterminate form $\frac{0}{0}$ as $x\to\infty$. We apply l'Hospital's rule:

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\left(\sin\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)'\cos\frac{1}{x}}{\left(\frac{1}{x}\right)'}$$

$$=\lim_{x\to\infty}\cos\frac{1}{x}=\cos 0=1. \blacktriangle$$

Example 4. Evaluate the limit

$$\lim_{x\to 0}\frac{\sqrt{x}}{\ln(1+x)}.$$

△ Since $\lim\limits_{x\to 0}\sqrt{x}=0$ and $\lim\limits_{x\to 0}\ln(1+x)=\ln 1=0$, there appears an indeterminate form $\frac{0}{0}$. The functions $y=\sqrt{x}$ and $y=\ln(1+x)$ differentiable on the interval $(0, +\infty)$ and continuous on the interval $[0, +\infty)$ (the function $y=\sqrt{x}$ is continuous at the point $x=0$

from the right), and the limit is evaluated as $x \to 0$. L'Hospital's rule holds for this case as well:

$$\lim_{x\to 0} \frac{\sqrt{x}}{\ln(1+x)} = \lim_{x\to 0} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\frac{1}{1+x}\right)} = \infty. \blacktriangle$$

Let us now consider another case of evaluating the limit of the ratio of two functions $u(x)$ and $v(x)$ tending to infinity as $x \to c$ (or as $x \to \infty$),

$$\lim_{x\to c} u(x) = \lim_{x\to c} v(x) = \infty. \tag{3}$$

In this case we speak of the indeterminate form $\frac{\infty}{\infty}$. For evaluating this indeterminate form, that is, for evaluating the limit $\lim\limits_{x\to c} \frac{u(x)}{v(x)}$, we apply formula (1) which is called l'Hospital's rule for evaluating indeterminate forms of the type $\frac{\infty}{\infty}$.

Example 5. Evaluate the limit

$$\lim_{x\to+\infty} \frac{x^2}{e^x}.$$

$\triangle$ Since $\lim\limits_{x\to+\infty} x^2 = +\infty$ and $\lim\limits_{x\to+\infty} e^x = +\infty$, it is impossible to apply the theorem on the limit of a quotient for evaluating the given limit since in this case there appears an indeterminate form $\frac{\infty}{\infty}$. We evaluate this indeterminate form using l'Hospital's rule, that is, applying formula (1):

$$\lim_{x\to+\infty} \frac{x^2}{e^x} = \lim_{x\to+\infty} \frac{(x^2)'}{(e^x)'} = \lim_{x\to+\infty} \frac{2x}{e^x}.$$

We have once again obtained an indeterminate form $\frac{\infty}{\infty}$, since, $\lim\limits_{x\to+\infty} (2x) = +\infty$ and $\lim\limits_{x\to+\infty} e^x = +\infty$. We evaluate it applying l'Hospital's rule:

$$\lim_{x\to+\infty} \frac{2x}{e^x} = \lim_{x\to+\infty} \frac{(2x)'}{(e^x)'} = \lim_{x\to+\infty} \frac{2}{e^x} = 0.$$

Thus, having applied formula (1) twice, we have found that

$$\lim_{x\to+\infty} \frac{x^2}{e^x} = 0. \blacktriangle$$

2. Indeterminate Forms of the Type $0\cdot\infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0. All these indeterminate forms are reduced to the above two principal types by identity transformations.

Example 1. Evaluate the limit

$$\lim_{x\to 0} x^2 e^{\frac{1}{x^2}}.$$

△ Since $\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0} e^{\frac{1}{x^2}} = +\infty$, there appears an indeterminate form $0\cdot\infty$. Using the identity transformation

$$x^2 e^{\frac{1}{x^2}} = \frac{e^{\frac{1}{x^2}}}{\left(\frac{1}{x^2}\right)},$$

we pass to the indeterminate form $\frac{\infty}{\infty}$, which is evaluated by applying l'Hospital's rule:

$$\lim_{x\to 0} x^2 e^{\frac{1}{x^2}} = \lim_{x\to 0} \frac{e^{\frac{1}{x^2}}}{\left(\frac{1}{x^2}\right)} = \lim_{x\to 0} \frac{\left(e^{\frac{1}{x^2}}\right)'}{\left(\frac{1}{x^2}\right)'}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{2}{x^3}\right) e^{\frac{1}{x^2}}}{\left(-\frac{2}{x^3}\right)} = \lim_{x\to 0} e^{\frac{1}{x^2}} = +\infty. \blacktriangle$$

Example 2. Evaluate the limit $\lim\limits_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)$.

△ Since $\lim\limits_{x\to 0} \frac{1}{\sin x} = \infty$ and $\lim\limits_{x\to 0} \frac{1}{\tan x} = \infty$, we have an indeterminate form $\infty - \infty$. With the help of the identity transformation

$$\frac{1}{\sin x} - \frac{1}{\tan x} = \frac{1-\cos x}{\sin x},$$

we pass to the indeterminate form $\frac{0}{0}$, and get by l'Hospital's rule:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{(1-\cos x)'}{(\sin x)'}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x} = 0. \blacktriangle$$

Example 3. Evaluate the limit $\lim\limits_{x\to 0} (1+x^2)^{\frac{1}{x^4}}$.

△ Since, as $x \to 0$, the base of the power tends to unity, while the exponent tends to infinity, there appears an indeterminate form

of the type 1^{∞}. With the help of the identity transformation

$$(1+x^2)^{\frac{1}{x^4}} = e^{\frac{\ln(1+x^2)}{x^4}}$$

we pass to the indeterminate form $\frac{0}{0}$ and using l'Hospital's rule find the limit:

$$\lim_{x\to 0}(1+x^2)^{\frac{1}{x^4}} = \lim_{x\to 0} e^{\frac{\ln(1+x^2)}{x^4}} = e^{\lim\limits_{x\to 0}\frac{\ln(1+x^2)}{x^4}}$$

$$= e^{\lim\limits_{x\to 0}\frac{\left(\frac{2x}{1+x^2}\right)}{4x^3}} = \lim_{x\to 0} e^{\frac{1}{2x^2(1+x^2)}} = +\infty,$$

where we have made use of the property of continuous functions:

$$\lim_{x\to c} f(\varphi(x)) = f\left(\lim_{x\to c}\varphi(x)\right). \blacktriangle$$

Example 4. Evaluate the limit $\lim\limits_{x\to+0} x^x$.

△ Since, as $x \to +0$, both the base of the power and exponent tend to zero, there appears an indeterminate form of the type 0^0. Using the identity transformation

$$x^x = e^{x\ln x} = e^{\frac{\ln x}{\left(\frac{1}{x}\right)}}$$

we pass to the indeterminate form $\frac{\infty}{\infty}$ and find the limit applying l'Hospital's rule:

$$\lim_{x\to+0} x^x = \lim_{x\to+0} e^{\frac{\ln x'}{\left(\frac{1}{x}\right)}} = e^{\lim\limits_{x\to+0}\frac{\ln x}{\left(\frac{1}{x}\right)}}$$

$$= e^{\lim\limits_{x\to+0}\frac{(\ln x)'}{\left(\frac{1}{x}\right)'}} = e^{\lim\limits_{x\to+0}(-x)} = e^0 = 1. \blacktriangle$$

Example 5. Evaluate the limit $\lim\limits_{x\to+\infty} x^{\frac{1}{x}}$.

△ Since, as $x \to +\infty$, the base of the power tends to infinity, while the exponent tends to zero, there appears an indeterminate form of the type ∞^0. With the help of the identity transformation

$$x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$$

we pass to the indeterminate form $\frac{\infty}{\infty}$ and, using l'Hospital's rule, find the limit:

$$\lim_{x\to+\infty} x^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{\ln x}{x}} = e^{\lim\limits_{x\to+\infty}\frac{\ln x}{x}} = e^{\lim\limits_{x\to+\infty}\frac{(\ln x)'}{x'}}$$

$$= e^{\lim\limits_{x\to+\infty}\frac{1}{x}} = e^0 = 1. \blacktriangle$$

Exercises

Using l'Hospital's rule, evaluate the following limits:

(a) $\lim\limits_{x\to 0}\frac{\ln\cos x}{x}$; (b) $\lim\limits_{x\to 0}\frac{e^x-1}{\sin x}$;

(c) $\lim\limits_{x\to 0}\frac{x-\arctan x}{x^3}$; (d) $\lim\limits_{x\to 0}\frac{e^{x^2}-1}{\cos x-1}$;

(e) $\lim\limits_{x\to+\infty}(x^4e^{-x})$; (f) $\lim\limits_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln x}\right]$;

(g) $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)$; (h) $\lim\limits_{x\to 0}\frac{\tan x}{\sin 2x}$;

(i) $\lim\limits_{x\to\infty}(1+x^2)^{\frac{1}{x^4}}$.

6.7. Applying the Derivative to Investigation of Functions

1. Monotonicity Intervals. Intervals on which a function decreases or increases are referred to as intervals of monotonicity of this function. Let us now clarify how we can judge the increase (decrease) of a function itself on a given interval from the derivative of the function. But first we shall consider the necessary condition.

Theorem 6.7.1 (*necessary condition*). *If a function* $f(x)$ *is differentiable and increases* (*decreases*) *on the interval* (a, b), *then*

$$f'(x) \geqslant 0 \quad (f'(x) \leqslant 0) \tag{1}$$

for any $x \in (a, b)$.

□ Let x_0 be some point belonging to the interval (a, b). Consider the case when the function $f(x)$ increases on the interval (a, b). Then for $x > x_0$ we have the inequality $f(x) \geqslant f(x_0)$, and for $x < x_0$ the inequality $f(x) \leqslant f(x_0)$. Consequently, for any $x \in (a, b)$, $x \neq x_0$ the following inequality is valid:

$$\frac{f(x)-f(x_0)}{x-x_0} \geqslant 0. \tag{2}$$

Since $f(x)$ is differentiable on (a, b), then, passing to the limit in inequality (2), as $x \to x_0$, we get

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0.$$

For the case of a decreasing function, the theorem is proved in a similar way. ■

When proving the theorem on the sufficient condition of a function to be monotonic, the following two theorems are used.

Theorem 6.7.2 (*Rolle's theorem*). *If a function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that*

$$f'(c) = 0. \tag{3}$$

□ As is known, a function continuous on a closed interval takes on the greatest and least values at certain points of this interval (see Chapter 5). Let M denote the greatest value of the function $f(x)$ on $[a, b]$, and m its least value. Then there appears the inequality $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. If $M = m$, then the function $f(x)$ is constant, and $f'(x) = 0$ for any $x \in (a, b)$.

Let now $M \neq m$. Then the condition $f(a) = f(b)$ implies that at least one of the numbers m and M is not attained at the end points of $[a, b]$. Let us assume, for definiteness, that such a number is M. Let the function take on its greatest value at the point $x = c \in (a, b)$ (that is, $f(c) = M$). Consequently, the inequality $f(x) - f(c) \leqslant 0$ is valid for all $x \in [a, b]$. But then we have the following inequalities:

$$\frac{f(x) - f(c)}{x - c} \leqslant 0$$

for $x > c$ and

$$\frac{f(x) - f(c)}{x - c} \geqslant 0$$

for $x < c$. Since, by the condition of the theorem, the derivative at the point $x = c$ exists, then, passing to the limit as $x \to c + 0$ and $x \to c - 0$, we get:

$$f'(c) = \lim_{x \to c+0} \frac{f(x) - f(c)}{x - c} \leqslant 0$$

and

$$f'(c) = \lim_{x \to c-0} \frac{f(x) - f(c)}{x - c} \geqslant 0.$$

Since both relations must be satisfied simultaneously, $f'(c) = 0$, and formula (3) has been proved. ■

Theorem 6.7.3 (*Lagrange's theorem*). *If a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is*

a point $c \in (a, b)$ *such that the following formula is valid:*

$$f(b) - f(a) = f'(c)(b - a). \tag{4}$$

□ Consider on $[a, b]$ the auxiliary function

$$F(x) = f(x) - x\,\frac{f(b)-f(a)}{b-a}.$$

It is easy to verify that the function $F(x)$ satisfies the conditions of Rolle's theorem. Indeed, the function $f(x)$ is continuous on the interval $[a, b]$, and the function $x\,\frac{f(b)-f(a)}{a-b}$, being a linear one, is continuous everywhere on the number line; therefore the function $F(x)$, as the difference of two continuous functions, is also continuous on $[a, b]$. The function $f(x)$ is differentiable on (a, b), and the function $x\,\frac{f(b)-f(a)}{b-a}$ is also differentiable at all the points on the number line, therefore the difference between these functions $F(x)$ is also differentiable on (a, b). Since

$$F(b) = F(a)\,\frac{b f(a) - a f(b)}{b-a},$$

all the conditions of Rolle's theorem are satisfied for the function $F(x)$. Therefore, according to Rolle's theorem, there is a point $c \in (a, b)$ such that

$$F'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0,$$

whence there follows the validity of formula (4). ■

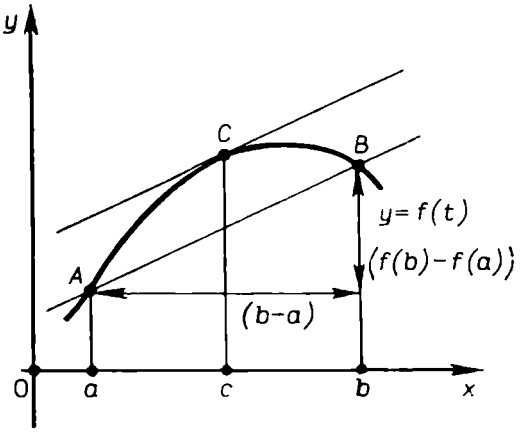

Fig. 102

Formula (4) is termed Lagrange's formula.

Let us now clear up the geometrical meaning of Lagrange's theorem. On the graph of the function $f(x)$, consider the points $A(a, f(a))$ and $B(b, f(b))$ and the secant AB joining these points (Fig. 102). Then the ratio $\frac{f(b)-f(a)}{b-a}$ is equal to the slope of the secant AB. We write formula (4) in the form

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \tag{5}$$

As is known, the geometrical meaning of the derivative of a function at a point consists in that the slope of the tangent line drawn to the graph of the function at this point coincides with the value of the derivative at the indicated point. Therefore it follows from formula (5) that the slope of the secant AB coincides with that of the

tangent to the graph of the function $f(x)$ at the point with abscissa c. Thus, Lagrange's theorem states that in the interval (a, b) there is a point at which the tangent to the graph of the function $f(x)$ is parallel to the secant AB.

Let us now formulate and prove the theorem on the sufficient condition for a function to increase.

Theorem 6.7.4 (*sufficient condition*). *If a function $f(x)$ has a positive (negative) derivative at every point of the interval (a, b), then this function strictly increases (strictly decreases) on the interval (a, b).*

□ Let x_1 and x_2 be two arbitrary points of the interval (a, b), where $x_1 < x_2$. According to Lagrange's theorem, there is a point $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1). \tag{6}$$

If $f'(x) > 0$, then for $x_2 > x_1$ formula (6) implies: $f(x_2) > f(x_1)$ which just means a strict increase in the function $f(x)$ on the interval (a, b). And if $f'(x) < 0$, then for $x_2 > x_1$ we find that $f(x_2) < f(x_1)$, that is, the function $f(x)$ strictly decreases. ■

Note. The condition formulated in Theorem 6.7.4 is sufficient for a function to be strictly increasing, but is not necessary. Indeed, for instance, the function $f(x) = \frac{x^3}{3}$ is strictly increasing throughout the number line; at the same time the derivative of this function $f'(x) = x^2$ vanishes at the point $x = 0$.

It follows from the above theorems that the intervals of monotonicity of the function $f(x)$ are separated from one another by points at which either the derivative $f'(x)$ vanishes (stationary points) or the derivative does not exist. These points are called critical for $f(x)$.

Let us now formulate the rule for finding the intervals of monotonicity of a function:

(1) Find the critical points of the function $f(x)$. The domain of definition of the function $f(x)$ is separated by these points into intervals on each of which the derivative $f'(x)$ retains its sign.

(2) Investigate the sign of $f'(x)$ on each of the obtained intervals. If on a certain interval $f'(x) > 0$, then the function $f(x)$ strictly increases on that interval, and where $f'(x) < 0$ it strictly decreases.

Example. Find the intervals of monotonicity of the function $f(x) = x \ln x + 3x$.

△ The given function is defined and differentiable at all points of the interval $(0, +\infty)$. We compute its derivative: $f'(x) = 1 + \ln x + 3 = 4 + \ln x$. From the equation $f'(x) = 4 + \ln x = 0$ we find the only critical point $x = e^{-4}$.

Since $f'(x) = 4 + \ln x < 0$ for $x \in (0, e^{-4})$, the given function strictly decreases on this interval. Since $f'(x) = 4 + \ln x > 0$ for

$x \in (e^{-4}, +\infty)$, the given function strictly increases on the interval $(e^{-4}, +\infty)$. ▲

2. The Extrema of a Function. Let us give the definition of the points of minimum and maximum of a function.

Definition 1. A point x_0 from the domain of definition of a function $f(x)$ is called the point of minimum (maximum) of this function if there is a δ-neighbourhood of the point x_0 such that for all $x \neq x_0$ from this δ-neighbourhood the inequality $f(x) > f(x_0)$ $(f(x) < f(x))_0$ is satisfied.

The points of maximum and minimum of a function are called the points of extremum of the function, and the values of the function at these points are respectively called the maximum and minimum of the function or the extrema of the function.

It may turn out that some maximum of a function is less than some minimum. This fact does not contradict the definitions of extrema of a function since in them the values of the function at a point are compared with the values of the function from a certain neighbourhood of this point. Thus, the concept of an extremum is always associated with a certain neighbourhood of the given point from the domain of definition of the function, but not with the entire domain. Therefore, for the definition of this concept the term "a local extremum" is sometimes used, that is, an extremum related with a certain location. Note that if a function $f(x)$ is given on the closed interval $[a, b]$, then the points a and b cannot be regarded as extremal points of the function $f(x)$ since they have no δ-neighbourhoods belonging to the domain of definition of the given function.

Let us now consider the necessary condition for an extremum of a differentiable function.

Theorem 6.7.5 (*Fermat's theorem*). *If x_0 is a point of extremum of a function $f(x)$, and the derivative $f'(x_0)$ exists at this point, then it is equal to zero*:

$$f'(x_0) = 0.$$

□ Let, for definiteness, x_0 be a point of maximum. Then, by definition, there exists a δ-neighbourhood of the point x_0 such that for all $x \neq x_0$ from this δ-neighbourhood the inequality $f(x) < f(x_0)$ is satisfied. By hypothesis, the function $f(x)$ has a derivative at the point x_0. Therefore

$$f'(x_0) = \lim_{x \to x_0 - 0} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0,$$

and, on the other hand,

$$f'(x_0) = \lim_{x \to x_0 + 0} \frac{f(x) - f(x_0)}{x - x_0} \leqslant 0.$$

Consequently, $f'(x_0) = 0$. ■

Let us now formulate the sufficient condition for existence of an extremum.

Theorem 6.7.6 (*sufficient condition for an extremum*). *Let a function* $f(x)$ *be continuous at a point* x_0 *and have a derivative everywhere in its* δ*-neighbourhood except perhaps at the point* x_0 *itself. Then*:

(a) *if, as* x *passes through the point* x_0, *the derivative* $f'(x)$ *changes sign from plus to minus, the point* x_0 *is a point of maximum of the function* $f(x)$;

(b) *if, as* x *passes through the point* x_0, *the derivative changes sign from minus to plus, the point* x_0 *is a point of minimum of the function* $f(x)$.

□ Let us prove the assertion (a) whose condition means that there is a number $\delta > 0$ such that $f'(x) > 0$ for all x from the interval $(x_0 - \delta, x_0)$, and $f'(x) < 0$ for all x from the interval $(x_0, x_0 + \delta)$. Then, by Theorem 6.7.4, the function $f(x)$ increases on the interval $(x_0 - \delta, x_0)$, and $f(x) < f(x_0)$ for all x from this interval, while on the interval $(x_0, x_0 + \delta)$ the function $f(x)$ decreases, so that $f(x) < f(x_0)$ for all x from this interval. Thus, the inequality $f(x) \leqslant f(x_0)$ is valid for all $x \neq x_0$ from the interval $(x_0 - \delta, x_0 + \delta)$, that is, according to the definition, the point x_0 is a point of maximum of the function $f(x)$. The assertion (b) is proved in a similar manner.

Let $f(x)$ be defined and continuous on the interval (a, b), have a derivative everywhere on (a, b) except perhaps at a finite number of points, and have no more than a finite number of stationary points. Then, in order to find the extrema of the function, it is necessary:

(1) to find the critical points of the function $f(x)$, that is, the points at which either $f'(x) = 0$ or $f'(x)$ is nonexistent;

(2) to investigate the sign of the derivative $f'(x)$ in some δ-neighbourhood of each critical point. And if, as x passes through a critical point, $f'(x)$ changes sign, then $f(x)$ has an extremum at this point. Namely, if minus is changed for plus, then the function has a minimum at this point; if the sign is altered from plus to minus, then a maximum is the case. And if the sign of $f'(x)$ remains unchanged as x passes through the point under consideration, then the function $f(x)$ has no extremum at this point.

Example. Find the extrema of the function $f(x) = x^2(x - 1)$, $x \in \boldsymbol{R}$.

△ We compute the derivative

$$f'(x) = 2x(x - 1) + x^2 = 3x^2 - 2x = x(3x - 2)$$

and find the stationary points: $x_1 = 0$ and $x_2 = \frac{2}{3}$. It is easy to see that $f'(x) > 0$ on the interval $(-\infty, 0)$, $f'(x) < 0$ on the interval $\left(0, \frac{2}{3}\right)$, and $f'(x) > 0$ on the interval $\left(\frac{2}{3}, +\infty\right)$. Consequently,

at the point $x_1 = 0$ the function $f(x)$ has a maximum ($f(0) = 0$), and at the point $x_2 = \frac{2}{3}$ a minimum $\left(f\left(\frac{2}{3}\right) = -\frac{4}{27}\right)$. ▲

Exercises

1. Find the intervals of monotonicity of the following functions:

(a) $f(x) = 5x - 2$; (b) $f(x) = 4 - 9x$; (c) $f(x) = \frac{1}{3x}$;

(d) $f(x) = \frac{4}{5-x}$; (e) $f(x) = x^2 + x - 1$; (f) $f(x) = (x+1)^3$;

(g) $f(x) = 7x^2 + 14x + 1$; (h) $f(x) = 3x^4 - 6x^2 + 4$;

(i) $f(x) = x(x^2 - 3)$; (j) $f(x) = x^3(1 - x)$;

(k) $f(x) = \frac{x}{x^2+1}$; (l) $f(x) = \frac{x^3}{1-x}$.

2. Find the extrema of the following functions:

(a) $f(x) = x^2 + 4x + 5$; (b) $f(x) = \frac{x}{2} + \frac{2}{x}$; (c) $f(x) = \frac{1}{1+x^2}$;

(d) $f(x) = \frac{1+\ln x}{x}$; (e) $f(x) = xe^{-\frac{x}{2}}$; (f) $f(x) = x \ln x$.

6.8. Constructing Graphs of Functions

1. Convexity of the Graph of a Function. Figure 103 represents the graph of a function $f(x)$ increasing on a closed interval $[a, b]$. We can easily note the following peculiarities in the behaviour of the graph of the given function: on the interval (a, c) the graph is "bulging upward", while on the interval (c, d) it is bulging downward. The point $x = c$ separates these intervals. Let us now attach a strict mathematical sense to these expressions.

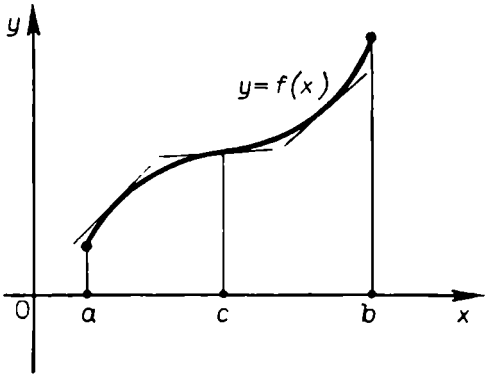

Fig. 103

Definition 1. The graph of a differentiable function $f(x)$, $x \in (a, b)$ is said to be convex upward on the interval (a, b) if the derivative $f'(x)$ strictly decreases on that interval. And if $f'(x)$ strictly increases on (a, b), then the graph of this function is said to be convex downward.

It is easy to see that if the graph is convex upward, then all of its points lie below any of its tangents (Fig. 103) since the slope of

a tangent decreases with an increase in x. And if on some interval the graph is convex downward, then all of its points lie above any of tangents (except, of course, for the point of tangency itself) drawn to the graph at any point of the interval since the slope of a tangent increases with an increase in x. A point at which the tangent to the graph of the function lies above the graph on one side, and below it on the other (that is, the tangent intersects the graph) is called the point of inflection (the point c in Fig. 103).

Definition 2. Intervals on which the graph of a function is convex upward or downward are called the intervals of convexity of the graph of a function.

Theorem 6.8.1 (*sufficient condition for convexity of a graph*). *Let a function* $f(x)$, $x \in (a, b)$, *have the first and second derivatives. Then, if* $f''(x) < 0$ *for all* $x \in (a, b)$, *the graph of the function* $f(x)$ *is convex upward on the interval* (a, b); *and if* $f''(x) > 0$ *for all* $x \in (a, b)$, *then the graph of the function* $f(x)$ *is convex downward on* (a, b).

□ As is known, if $f'' = (f'(x))' < 0$ for all $x \in (a, b)$, then the function $f'(x)$ strictly decreases on the interval (a, b) (see Theorem 6.7.4). Therefore, according to Definition 1, the graph of the function $f(x)$ will be convex upward on the interval (a, b). If now $f'' = (f'(x))' > 0$ for all $x \in (a, b)$, then $f'(x)$ strictly increases on the interval (a, b) (see Theorem 6.7.4). Consequently, according to Definition 1, the graph of the function $f(x)$ will be convex downward on the interval (a, b). ■

Let us now formulate the rule for finding the intervals of convexity and the points of inflection of a graph. Let a function $f(x)$ have a second derivative everywhere on the interval (a, b) except, perhaps, a finite number of points, and let $f''(x)$ have no more than a finite number of zeros in the interval (a, b). Then, in order to find the intervals of convexity and the points of inflection of the graph, we have:

(1) to find all the points at which either $f''(x) = 0$ or $f''(x)$ does not exist (these points are referred to as the critical points of the function with respect to the second derivative);

(2) to determine the sign of $f''(x)$ in each of the subintervals into which the interval (a, b) is separated by the above found points;

(3) if $f''(x) > 0$, then the graph of the function is convex downward on this interval, and if $f''(x) < 0$, then it is convex upward;

(4) if, as the argument x passes through a critical point $x_0 \in (a, b)$, the second derivative $f''(x)$ changes sign, then the point with coordinates $(x_0, f(x_0))$ is a point of inflection of the graph of the function $f(x)$, $x \in (a, b)$.

Example. Find the intervals of convexity and the points of inflection of the function $f(x) = x^4 - 2x^3 + 2$.

△ The given function has the derivatives $f'(x) = 4x^3 - 6x^2$ and $f''(x) = 12x(x-1)$ everywhere on the number line. The critical points of the function with respect to the second derivative are found from the equation $12x(x-1) = 0$, whence $x_1 = 0$ and $x_2 = 1$. We then determine the sign of $f''(x)$ on each interval: $f''(x) > 0$ on the interval $(-\infty, 0)$, $f''(x) < 0$ on the interval $(0, 1)$, and $f''(x) > 0$ on the interval $(1, +\infty)$. Consequently, the graph is convex downward on the intervals $(-\infty, 0)$ and $(1, +\infty)$ and it is convex upward on the interval $(0, 1)$. The points of the graph with coordinates $(x_1, f(x_1)) = (0, 2)$ and $(x_2, f(x_2)) = (1, 1)$ are points of inflection. ▲

2. Asymptotes to the Graph of a Function. A straight line $y = kx + b$ is called an oblique asymptote to the graph of a function $f(x)$ as $x \to \pm\infty$ if

$$\lim_{x\to\pm\infty} (f(x) - kx - b) = 0, \tag{1}$$

that is, the function

$$\alpha(x) = f(x) - kx - b \tag{2}$$

is infinitely small as $x \to \pm\infty$. It follows from formula (2) that

$$k = \frac{f(x)}{x} - \frac{b + \alpha(x)}{x}.$$

Passing to the limit in the obtained equality, as $x \to \pm\infty$, we find:

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x}, \tag{3}$$

since $\lim\limits_{x\to\pm\infty} \dfrac{b+\alpha(x)}{x} = 0$. Similarly, we find that

$$b = \lim_{x\to\pm\infty} (f(x) - kx). \tag{4}$$

In the particular case $k = 0$ the asymptote is called horizontal. The parameters of an oblique asymptote k and b are computed by formulas (3) and (4). In the general case, the asymptotes to the graph of the function $f(x)$ as $x \to +\infty$ and $x \to -\infty$ are different. If finite limit (3) does not exist, then the graph has no oblique asymptotes.

Example 1. Find the asymptotes to the graph of the function $f(x) = \dfrac{2x^2-1}{x}$.

△ Using formulas (3) and (4), we compute:

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{2x^2-1}{x^2} = 2;$$

$$b = \lim_{x\to+\infty} [f(x) - kx] = \lim_{x\to+\infty} \left(\frac{2x^2-1}{x} - 2x\right) = \lim_{x\to+\infty} \left(-\frac{1}{x}\right) = 0.$$

The straight line $y = 2x$ is an oblique asymptote to the graph of the function $f(x)$ as $x \to +\infty$. Obviously, the same straight line is an asymptote as $x \to -\infty$. ▲

Example 2. Find the asymptotes to the graph of the function $f(x) = xe^x$.

△ We compute:

$$k = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} e^x = +\infty,$$

and the graph has no oblique asymptote as $x \to +\infty$. For $x \to -\infty$ we have:

$$k = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} e^x = 0;$$

$$b = \lim_{x \to -\infty} [f(x) - kx] = \lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}}$$

$$= \lim_{x \to -\infty} \frac{(x)'}{(e^{-x})'} = \lim_{x \to -\infty} \frac{1}{(-e^{-x})} = -\lim_{x \to -\infty} e^x = 0.$$

Thus, the graph has a horizontal asymptote $y = 0$ as $x \to -\infty$. ▲

Definition. A straight line $x = c$ is called a vertical asymptote to the graph of a function $f(x)$ if

$$\lim_{x \to c-0} f(x) = \infty \text{ or } \lim_{x \to c+0} f(x) = \infty.$$

Note that in finding vertical asymptotes to the graph of a function $f(x)$, it is advisable to consider the points of discontinuity of the second kind of the given function $f(x)$ as points c.

Example 3. Find the vertical asymptotes to the graph of the function $f(x) = \frac{2}{x^2 - 1}$.

△ The denominator vanishes at the points $x_1 = -1$ and $x_2 = 1$. We have

$$\lim_{x \to \pm 1} f(x) = \lim_{x \to \pm 1} \frac{2}{x^2 - 1} = \infty,$$

hence the straight lines $x = -1$ and $x = 1$ are vertical asymptotes. ▲

3. Sketching the Graphs of Functions. For this purpose, the following scheme is suggested:

(1) Find the domain of definition of a function.

(2) Check to see whether the function is even or odd.

(3) Investigate the given function for periodicity.

(4) Find the points at which the graph of the function intersects the coordinate axes.

(5) Find the intervals of constant signs of the function.

(6) Find the asymptotes (oblique and vertical) to the graph of the function.

(7) Find the intervals of monotonicity and the points of extremum of the function.

(8) Find the points of inflection and intervals of convexity of the graph of the function.

(9) Sketch the graph of the function.

Example. Construct the graph of the function $f(x)=\frac{x^3}{1-x^2}$.

△ We proceed according to the scheme:

(1) The domain of definition of the function is the entire number line except for the points $x=1$ and $x=-1$.

(2) The given function is odd since $f(-x)=-f(x)$. Therefore, to construct the graph of the function $y=f(x)$, it suffices to investigate it for $x \geqslant 0$.

(3) The function is nonperiodic.

(4) The graph of the function intersects the coordinate axes at the point (0, 0).

(5) $f(x)>0$ on the interval (0, 1), and $f(x)<0$ on the interval $(1, +\infty)$.

(6) Since the denominator vanishes at the points $x=1$ and $x=-1$, we have

$$\lim_{x \to 1 \pm 0} f(x)=\pm \infty \text{ and } \lim_{x \to -1 \pm 0} f(x)=\pm \infty,$$

and the straight lines $x=1$ and $x=-1$ are vertical asymptotes. Since

$$\lim_{x \to \pm \infty} \frac{f(x)}{x}=\lim_{x \to \pm \infty} \frac{x^2}{1-x^2}=-1,$$

$$\lim_{x \to \pm \infty} [f(x)-kx]=\lim_{x \to \pm \infty} \left(\frac{x^3}{1-x^2}+x\right)=0,$$

there is an oblique asymptote $y=-x$ for $x \to \pm \infty$.

(7) The derivative

$$f'(x)=\frac{x^2(3-x^2)}{(1-x^2)^2}$$

exists at all points of the number line except for $x=\pm 1$, and is equal to zero at the points $x=0$ and $x=\pm \sqrt{3}$. Thus, the function has the following critical points:

$$x_1=-\sqrt{3}, \quad x_2=-1, \quad x_3=0, \quad x_4=1, \quad x_5=\sqrt{3}.$$

Let us study the behaviour of $f'(x)$ in the neighbourhood of each critical point. Since $f(x)$ is odd, it suffices to consider the sign of

$f'(x)$ on the intervals $(-1, 0)$, $(0, 1)$, $(1, \sqrt{3})$, and $(\sqrt{3}, +\infty)$. The results are then tabulated

Table 1

x	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$1 < x < \sqrt{3}$	$x = \sqrt{3}$	$\sqrt{3} < x < +\infty$
$f'(x)$	+	0	+	+	0	−
$f(x)$	Increases	No extremum	Increases	Increases	Maximum	Decreases

The function has no extremum at the points $x = \pm 1$ since they do not belong to the domain of definition of the given function.

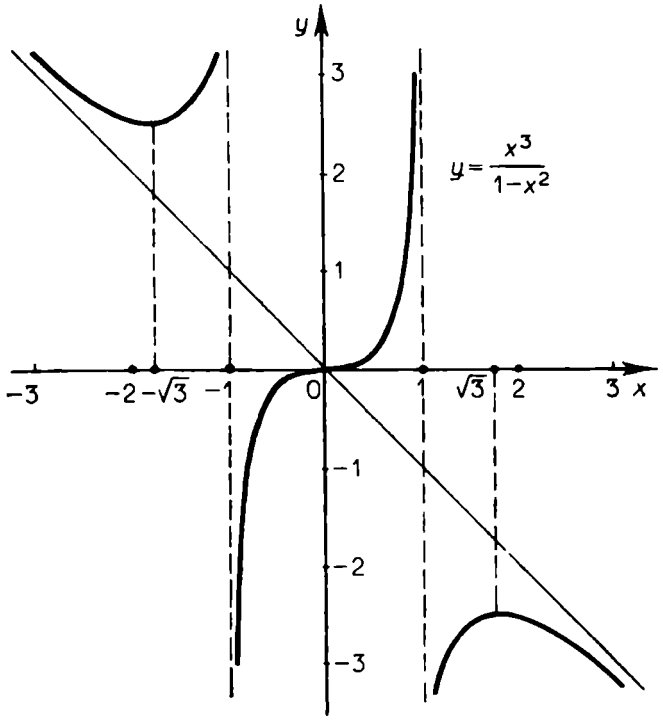

Fig. 104

Since the function is odd, we may assert that it has a minimum for $x = -\sqrt{3}$.

(8) In order to investigate the graph of the given function for convexity, let us find the second derivative:

$$f''(x) = \frac{2x(x^2+3)}{(1-x^2)^3}.$$

The critical points of the given function with respect to the second derivative consist of the root of the equation $f''(x) = 0$, that is, $x = 0$ and the points where $f''(x)$ does not exist, that is, $x = \pm 1$. But the points $x = \pm 1$ do not belong to the domain of definition of the function, therefore a point of inflection is expected for sure only at the point with abscissa $x = 0$. We then investigate the sign of the second derivative and tabulate the results:

Table 2

x	$x < -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x > 1$
$f''(x)$	+	−	0	+	−
$f(x)$	Convex downward	Convex upward	Point of inflection	Convex downward	Convex upward

(9) On the basis of the results of the above investigation of the given function, we sketch its graph (Fig. 104). ▲

Exercises

1. Find the intervals on which the graphs of the given functions are convex upward or downward:

(a) $f(x) = x^3 - 6x^2 + 12x + 4$;

(b) $f(x) = (x + 1)^4$;

(c) $f(x) = x^4 - 6x^2 + 4$;

(d) $f(x) = x^4 + 8x^2 + 16$.

2. Investigate the quadratic functions and sketch their graphs:

(a) $f(x) = x^2 + 2x - 3$; (b) $f(x) = 4x^2 - 6x - 7$;

(c) $f(x) = 3 + 4x - x^2$; (d) $f(x) = x^2 - 2x + 3$.

3. Find the asymptotes to the graphs of the given functions:

(a) $y = \frac{1}{2-x}$; (b) $y = \frac{x^3}{1-x}$; (c) $y = \sqrt{x^2-4}$;

(d) $y = \frac{x^2-5x+6}{x}$; (e) $y = \frac{x^4}{(1-x)^3}$; (f) $y = \frac{1}{1+x^2}$.

4. Investigate the given functions and sketch their graphs:

(a) $f(x) = x^3 - 3x^2 - x + 3$; (b) $f(x) = x^4 - 10x^2 + 9$;

(c) $f(x) = \frac{x-1}{(2x+1)(2-x)}$; (d) $f(x) = \frac{(x+1)(2-x)}{2x+3}$;

(e) $f(x)=\dfrac{1}{x^2-5x+6}$; (f) $f(x)=\dfrac{x^2-4}{x^2-16}$;

(g) $f(x)=3^{1/x}$; (h) $f(x)=x\ln x$;

(i) $f(x)\,e^{\sin x}$; (j) $f(x)=e^{1/x}$.

6.9. Solving Problems on the Greatest and Least Values of Functions

1. The Greatest and Least Values of a Function on a Closed Interval. If a function $f(x)$ is continuous on a closed interval $[a, b]$, then among its values on this interval there are both the greatest and the least value. Let, in addition, the function $f(x)$ be differentiable on the interval (a, b) except, perhaps, a finite number of points. The greatest and least values of the function $f(x)$ on the closed interval $[a, b]$ can be attained by the function at one of the critical points or at one of the endpoints. Let us formulate the rule for finding the greatest (least) value of the function $f(x)$ on the closed interval $[a, b]$:

(1) find the critical points of the function $f(x)$ on the open interval (a, b);

(2) evaluate the function $f(x)$ at the critical points;

(3) evaluate the function at the endpoints of $[a, b]$, that is, find $f(a)$ and $f(b)$;

(4) among the found values of the function, choose the greatest (least) value.

Example 1. Find the greatest and least values of the function $f(x)=2x^3-9x^2+12x-3$ on the closed interval $[0, 3]$.

△ Solving the equation $f'(x)=6x^2-18x+12=6(x-1)\times(x-2)=0$, we find the critical points $x_1=1$ and $x_2=2$. According to the above rule, the largest of the numbers

$$f(0)=-3,\quad f(1)=2,\quad f(2)=1,\quad f(3)=6$$

will be the greatest value of the given function on $[0, 3]$, and the smallest of these numbers the least value of the given function on the interval $[0, 3]$. Thus, $f_{\text{greatest}}=f(3)=6$ and $f_{\text{least}}=f(0)=-3$. ▲

Example 2. Find the greatest and least values of the function $f(x)=\sqrt[3]{x}\,(x-2)$ on the closed interval $[-1, 1]$.

△ The function $f(x)$ is continuous on $[-1, 1]$.

(1) Its derivative

$$f'(x)=\frac{1}{3(\sqrt[3]{x})^2}(x-2)+\sqrt[3]{x}=\frac{2(2x-1)}{3(\sqrt[3]{x})^2}$$

does not exist (has no finite value) at the point $x_1=0$ and vanishes at the point $x_2=\dfrac{1}{2}$.

(2) The values of the function at the critical points are equal to:

$$y_1 = f(0) = 0, \quad y_2 = f\left(\frac{1}{2}\right) = -\frac{3}{2\sqrt[3]{2}}.$$

(3) At the endpoints of the closed interval the function $f(x)$ takes on the values:

$$y_3 = f(-1) = 3, \quad y_4 = f(1) = -1.$$

(4) Of the found values of the function y_1, y_2, y_3, y_4 the greatest is y_3, and the least y_2. Thus, $f_{\text{greatest}} = y_3 = f(-1) = 3$ and $f_{\text{least}} = y_2 = f\left(\frac{1}{2}\right) = -\frac{3}{2\sqrt[3]{2}}$. ▲

2. Finding the Greatest and Least Values in Geometrical and Physical Problems. There are many geometrical and physical problems among those with concrete contents whose solution requires finding the greatest or the least value of a certain function.

Example 1. Find the ratio of the altitude of a right circular cylinder to the diameter of its base for which, with the volume V fixed, the surface area S will be the least.

△ We denote the radius of the base of the cylinder by r, and the altitude of the cylinder by H. Then $S = 2\pi r^2 + 2\pi rH$. Since $V = \pi r^2 H$, we have $H = \frac{V}{\pi r^2}$ and $S = 2\left(\pi r^2 + \frac{V}{r}\right)$ is a function of one variable r. Let us find the least value of this function on the open interval $(0, +\infty)$. Since

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V}{r^2}\right),$$

there is only one critical (stationary) point on the interval $(0, +\infty)$ $r_1 = \sqrt[3]{\frac{V}{2\pi}}$, at which just the least value is attained since $\lim\limits_{r \to +0} S(r) = +\infty$ and $\lim\limits_{r \to +\infty} S(r) = +\infty$. Hence $H_1 = \frac{V}{\pi r_1^2} = 2\sqrt[3]{\frac{V}{2\pi}} = 2r_1$. Thus, for the surface area S of the cylinder to be the least at a given volume V, the altitude of the cylinder must be equal to its diameter. ▲

Example 2. Which of the rectangles having the perimeter P has the greatest area?

△ Let x be the length of one of the sides. Then the length of the other side is $\frac{(P-2x)}{2}$, and the area $S(x) = x\frac{(P-2x)}{2}$, $x \in \left[0, \frac{P}{2}\right]$. Since $S' = \frac{P}{2} - 2x$, we have $x_1 = \frac{P}{4}$ which is the only critical (stationary) point of this function. We have $S(0) = 0$,

$S\left(\frac{P}{4}\right)=\frac{P^2}{16}$, $S\left(\frac{P}{2}\right)=0$. Thus, with the perimeter P fixed, the square has the greatest area: $S_{\text{greatest}}=\frac{P^2}{16}$. ▲

Example 3. A yard-keeper hoses a lawn. The outflow velocity of water from the hose is v_0. At what angle φ to the horizon must he keep the hose to ensure the maximal hosing range?

△ The problem on computing the flying range of a body thrown at an angle φ to the horizon with an initial velocity v_0 is solved in the school course of physics. Let us recall the main steps of solution. The body (water particles) moves in a parabolic path so that the horizontal projection of the velocity vector is constant and is equal to $v_0 \cos \varphi$. Consequently, the flying range (for water it is the hosing range) L is computed by the formula

$$L = v_0 \cos \varphi \cdot T, \tag{1}$$

where T is the total time of flight which is twice the time necessary for the water particles to reach the upper point of the trajectory. The projection v_y of the velocity vector v on the vertical direction depends on time t according to the law

$$v_y = v_0 \sin \varphi - gt, \tag{2}$$

where g is free fall acceleration. Since at the upper point of the trajectory $v_y = 0$, from formula (2) we find the climbing time t_{climb}:

$$t_{\text{climb}}=\frac{v_0 \sin \varphi}{g}.$$

Substituting the time of flight $T=2t_{\text{climb}}=\frac{2v_0 \sin \varphi}{g}$ into formula (1), we find the final formula for the hosing range:

$$L=\frac{v_0^2 \sin 2\varphi}{g},$$

where $\varphi \in \left[0, \frac{\pi}{2}\right]$. In order to find the maximal hosing range, we find the critical points of the function $L=L(\varphi)$. We compute the derivative

$$L'(\varphi)=\frac{2v_0^2 \cos 2\varphi}{g}=0,$$

whence $\varphi_1=\frac{\pi}{4}$ is the only critical (stationary) point. Among the values $L(0)=0$, $L\left(\frac{\pi}{4}\right)=\frac{v_0^2}{g}$, $L\left(\frac{\pi}{2}\right)=0$ the greatest value $L_{\text{greatest}}=L\left(\frac{\pi}{4}\right)=\frac{v_0^2}{g}$ is attained when hosing the lawn at an angle $\frac{\pi}{4}$ radians (or 45°) to the horizon. ▲

Example 4. To a body of weight P lying on a rough horizontal surface a force F is applied at an angle φ to the horizon (Fig. 105). Find the angle φ at which the body can be shifted by the least (in magnitude) force F. The frictional force is computed by the formula $F_{fr} = kN$, where k is a frictional coefficient, and N the support reaction force.

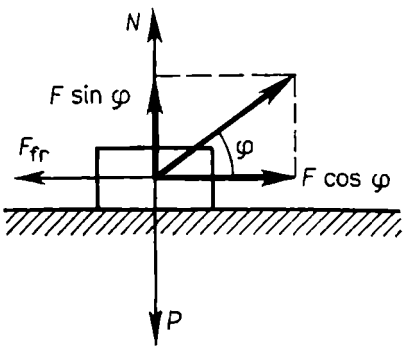

Fig. 105

△ The body will start moving if the horizontal component of the force $F \cos \varphi$ will be greater than, or equal to, the frictional force F_{fr} (if the forces are equal, then an additional initial push is needed to impart to the body a certain velocity which will be retained all the time according to Newton's first law). Thus, the minimal force corresponds to the equality

$$F \cos \varphi = F_{fr} = kN.$$

Since $N + F \sin \varphi = P$ (the body rests along the vertical axis), we have

$$F \cos \varphi = kP - kF \sin \varphi,$$

whence

$$F(\varphi) = \frac{kP}{\cos \varphi + k \sin \varphi},$$

where $\varphi \in \left[0, \frac{\pi}{2}\right]$ according to the sense of the problem.

We now determine for what angle φ the value of the shifting force F will be the least. For this purpose, we compute the derivative

$$F'(\varphi) = \frac{kP(\sin \varphi - k \cos \varphi)}{(\cos \varphi + k \sin \varphi)^2}$$

and find the only critical (stationary) on $\left[0, \frac{\pi}{2}\right]$ point $\varphi_1 = \arctan k$. Of the values

$$F(0) = kP, \quad F(\varphi_1) = \frac{kP}{\sqrt{1+k^2}} \left(\text{since } \cos \varphi_1 = \frac{1}{\sqrt{1+\tan^2 \varphi_1}} = \frac{1}{\sqrt{1+k^2}}\right),$$

$$F\left(\frac{\pi}{2}\right) = P$$

the least value $F_{\text{least}} = \frac{kP}{\sqrt{1+k^2}}$ is attained at an angle of inclination $\varphi = \varphi_1 = \arctan k$ to the horizon. ▲

Exercises

1. Find the greatest and least values of the given functions:

(a) $f(x)=x^3-3x$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$;

(b) $f(x)=x^4-8x^2-9$ on the intervals $[-1, 1]$, $[0, 3]$, and $[-3, 5]$;

(c) $f(x)=\sqrt{x^2}(2-x)$ on the intervals $[-6, -1]$ and $[-2, 1]$.

2. It is required to make a box in the shape of a parallelepiped whose volume must equal V. The box must have a square bottom and no cover. What dimensions must the box have to ensure the least amount of the material needed for its manufacture?

3. A rectangular plot adjacent to a wall is to be fenced with a wire-net of length L. Find the dimensions of the plot for which its area will be the greatest.

4. Prove that of all isosceles triangles inscribed in a given circle the equilateral triangle has the greatest perimeter.

5. Find the angle at the vertex of an isosceles triangle of a given area S for which the radius of the circle inscribed in this triangle will be the greatest.

6. A rain drop of initial mass m falls freely in the gravitational field of the Earth evaporating uniformly (the mass of the drop decreases proportional to time: $\Delta m = k \cdot \Delta t$). Find the instant of time when the kinetic energy of the drop will be the greatest.

7. A lamp hangs above the centre of a round table of radius R. At what height of the lamp above the table will the illumination of an object lying at the edge of the table be the best? (*Hint*: illumination is directly proportional to the cosine of the angle of incidence of beams and inversely proportional to the square of the distance from a light source).

8. The energy yielded by a source of direct current is equal to $W=\frac{E^2R}{(r+R)^2}$, where E is the electromotive force of the source, r the internal resistance, and R the load impedance. Find R for which the energy yielded by the source will be the greatest.

9. Find the external resistance R for which the source efficiency $\eta=\frac{W}{W+W_i}$, will be maximal, where W is determined in the preceding problem, and $W_i=\frac{E^2r}{(r+R)^2}$.

6.10. The Differential of a Function

1. The Differential of a Function Defined. According to the definition of the derivative of a function $f(x)$ at a point x_0, we have

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Using the property of the limit, we write this equality in the form

$$\frac{\Delta f(x_0)}{\Delta x} = f'(x_0) + \alpha(\Delta x), \tag{1}$$

where $\alpha(\Delta x) \to 0$ as $\Delta x \to 0$, and $\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0)$ is the increment in the function at the point x_0 corresponding to the increment in the argument Δx. From formula (1) we find that

$$\Delta f(x_0) = f'(x_0)\,\Delta x + \alpha(\Delta x)\cdot\Delta x. \tag{2}$$

Thus, if the function $f(x)$ has a nonzero derivative at the point x_0, then the increment in the function at this point consists of two terms. The first term in formula (2) is a linear function of Δx which is an infinitesimal as $\Delta x \to 0$. The second term in the same formula is also an infinitesimal as $\Delta x \to 0$ such that the ratio of the second term to the first tends to zero as $\Delta x \to 0$. Thus, the first term in formula (2) is the principal part of the increment in the function at x_0 for $f'(x_0) \neq 0$.

Definition. If a function $f(x)$ has a derivative $f'(x_0)$ at a point x_0, then the product $f'(x_0)\,\Delta x$ is called the differential of the function $f(x)$ at the point x_0 and is denoted by $df(x_0)$.

Thus, by definition

$$df(x_0) = f'(x_0)\cdot\Delta x.$$

For a function $f(x)$, having a derivative at every point of the interval (a, b), we may write:

$$df(x) = f'(x)\cdot\Delta x,$$

where Δx is an arbitrary increment in the argument.

Noting that $dx = (x)'\cdot\Delta x = \Delta x$, we define the differential of an independent variable as its increment. We then get the following expression for the differential of the function $f(x)$:

$$df(x) = f'(x)\cdot dx. \tag{3}$$

It follows from formula (3) that

$$f'(x) = \frac{df(x)}{dx},$$

that is, the designation $\frac{df(x)}{dx}$ for the derivative of the function $f(x)$ may be understood as a common fraction whose numerator is the differential of the function $f(x)$, and denominator is the differential of the argument.

Formula (3) enables us to evaluate the differentials of functions, provided that their derivatives are known. For instance, $dc = (c)'\cdot dx = 0\cdot dx = 0$, where c is a constant.

Example. Compute the differentials of the given functions:

(a) $dx^2 = (x^2)'dx = 2xdx$;

(b) $d(5x^4 + 7x - 2) = (5x^4 + 7x - 2)'dx = (20x^3 + 7)\,dx$;

(c) $d(\sin x + x) = (\sin x + x)'dx = (\cos x + 1)\,dx$;

(d) $d(e^x + \cos 2x) = (e^x + \cos 2x)'\,dx = (e^x - 2\sin 2x)\,dx$;

(e) $d \ln x = (\ln x)'\,dx = \dfrac{dx}{x}$.

For differentiable functions $u = u(x)$ and $v = v(x)$ the following equalities are valid:

$$d(u \pm v) = du \pm dv,$$

$$d(uv) = udv + vdu,$$

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

□ Let us prove, for example, the first of these equalities. By the definition of the differential, we have

$$d(u \pm v) = (u \pm v)'dx = u'dx \pm v'dx = du \pm dv.$$

The rest of the equalities are proved in a similar manner. ■

The differential of the composite function $Z(x) = \varphi(g(x))$ composed from the functions $Z = \varphi(y)$ and $y = g(x)$ is computed by the formula

$$dZ = Z'(x)\,dx = \varphi'(y)\,g'(x)\,dx = \varphi'(y)\,dy,$$

where we have made use of the equality $dy = g'(x)\,dx$ and the rule for differentiating a composite function. Thus, the differential of a composite function has the form of the product of the derivative with respect to a certain variable by the differential of the same variable both in the case when this variable is independent and when it is, in turn, a function. This property of the differential is called the invariance property of its form with respect to the choice of variables.

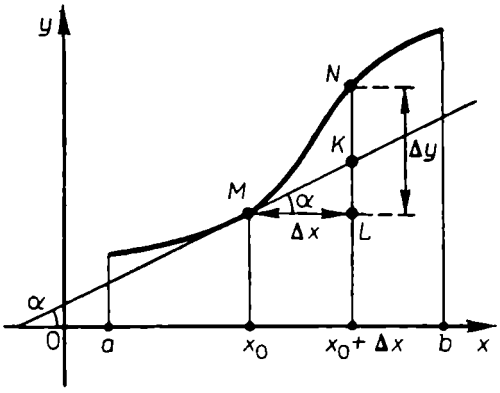

Fig. 106

2. Geometrical Meaning of the Differential. Consider the graph of the differentiable function $y = f(x)$, $x \in [a, b]$ and the tangent line MK drawn at the point $M(x_0, f(x_0))$ (Fig. 106). From the triangle MKL we have

$$|KL| = |ML| \cdot \tan\alpha.$$

Since $|ML| = \Delta x$ and $\tan \alpha = f'(x_0)$, we have $|KL| = f'(x_0) \cdot \Delta x$, that is $|KL| = dy$. The last equality allows us to give the following geometrical interpretation of the differential: if a function $f(x)$ has a derivative at a point x_0, then the differential at the point x_0 is equal to the increment in the ordinate of a tangent line drawn to the graph of the given function at the point with abscissa x_0 when passing from the point of tangency to the point with abscissa $x_0 + \Delta x$.

Generally speaking, the differential is not equal to the increment in the function:

$$dy \neq \Delta y,$$

since $|KL| \neq |NL|$. However, at small values of Δx the increment of a function is approximately equal to the differential of this function, that is, $\Delta y \approx dy$. This relation is used for computing the increment in a function with a small error.

3. Applying the Differential to Approximate Calculations. For a function $f(x)$, differentiable at a point x_0 and having $f'(x_0) \neq 0$, for sufficiently small Δx the following approximate equality is valid:

$$\Delta f(x_0) \approx df(x_0) = f'(x_0) \cdot \Delta x, \tag{1}$$

both the absolute error of this approximation $\Delta f(x_0) - df(x_0) = \alpha(\Delta x) \cdot \Delta x$ and its relative error

$$\frac{\Delta f(x_0) - df(x_0)}{df(x_0)} = \frac{\alpha(\Delta x) \cdot \Delta x}{f'(x_0) \cdot \Delta x} = \frac{\alpha(\Delta x)}{f'(x_0)}$$

tending to zero as $\Delta x \to 0$. Formula (1) is the base for simple approximate calculations. It implies that

$$f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0), \tag{2}$$

where we have set $\Delta x = x - x_0$ and taken into account that $\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = f(x) - f(x_0)$.

Example 1. Compute $\sqrt{3.998}$.

△ Direct computation of the root is difficult. Consider the function $f(x) = \sqrt{x}$, $x \in (0, +\infty)$. For this function formula (2) takes the form:

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} \cdot (x - x_0).$$

Substituting $x = 3.998$ and $x_0 = 4$, we find:

$$\sqrt{3.998} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(3.998 - 4) = 2 - \frac{0.002}{4} = 1.9995. \ \blacktriangle$$

Example 2. Compute $\sqrt[5]{243.45}$.

△ For the function $f(x)=\sqrt[5]{x}$, $x\in\mathbf{R}$, formula (2) takes the form:

$$\sqrt[5]{x}\approx\sqrt[5]{x_0}+\frac{x-x_0}{5x_0^{4/5}}.$$

Substituting $x=243.45$ and $x_0=243=3^5$, we find:

$$\sqrt[5]{243.45}\approx 3+\frac{0.45}{5.81}\approx 3.001. \blacktriangle$$

Exercises

Compute approximately:

(a) $\sqrt{9.02}$; (b) $\sqrt[3]{3}$; (c) $\sqrt[3]{24}$; (d) $\sqrt[3]{30}$;
(e) $\sqrt[4]{3}$; (f) $\sqrt[4]{90}$; (g) $\sqrt[3]{65}$; (h) $\sqrt[10]{1000}$;
(i) $\sqrt{125.1324}$; (j) $\sin 29°$; (k) $\ln(1.05)$; (l) $\tan 44°$.

6.11. Taylor's Formula

1. Taylor's Formula with the Remainder in Lagrange's Form. We saw in the preceding section that if a function $f(x)$ is differentiable at a point x_0, then in the neighbourhood of this point it can be approximately computed by the formula

$$f(x)\approx f(x_0)+f'(x_0)\cdot(x-x_0), \tag{1}$$

both the absolute and relative errors of this approximation tending to zero as $x\to x_0$. Consider, for instance, the function $f(x)=\sin x$ differentiable everywhere on the number line. We choose $x_0=0$. Then formula (1), enabling us to compute approximately the values of $\sin x$ for the arguments x close to zero, takes the form

$$\sin x\approx x. \tag{2}$$

It is seen from Table 1 that with an increase in x the accuracy of formula (2) keeps worsening.

Table 1

x	Value of $\sin x$ accurate to two decimal digits	Approximate value $\sin x\approx x$
0	0	0
$\frac{\pi}{12}$	0.26	0.26
$\frac{\pi}{6}$	0.50	0.52
$\frac{\pi}{4}$	0.71	0.79
$\frac{\pi}{3}$	0.87	1.05
$\frac{\pi}{2}$	1.00	1.57

In order to continue for large x the "operation" of formula (1), convenient for computing $\sin x$, we may set in it $x_0 = \frac{\pi}{4}$:

$$\sin x \approx \frac{\sqrt{2}}{2}\left(1 + x - \frac{\pi}{4}\right),$$

but thereby we have worsened accuracy for small x. There arises here an actual question: Is it possible to generalize formula (1) in order to broaden the domain of permissible deviations of x from x_0? The answer to this question is provided by the following theorem given without proof.

Theorem 6.11.1. *Let a function $f(x)$ be defined on an interval (a, b) and have at a point $x_0 \in (a, b)$ derivatives up to the $(n+1)$st order inclusively. Then the following formula is valid:*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x), \tag{3}$$

where

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \tag{4}$$

the point c lying between x and x_0.

Formula (3) is called Taylor's formula of the nth order with the remainder $r_n(x)$ in Lagrange's form. The polynomial

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n}(x - x_0)^n$$

is called Taylor's polynomial.

The above considered formula for apppoximate calculations (1) is a particular case of the more exact formula

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{5}$$

To illustrate this, let us consider the function $f(x) = \sin x$ once again and set $x_0 = 0$ and $n = 5$ in formula (5):

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}.$$

For $x = \frac{\pi}{6}$ the computations by this formula lead to the result $\sin x \approx 0.50$ which is more accurate than the result of the computations by formula (2) (see Table 1).

The possibility of approximate calculations by formula (5) for large deviations of x from x_0 is based on dependence of the function $r_n(x)$ on the order of Taylor's formula n. It is seen from formula

(4) that the remainder can be small even for a large deviation of x from x_0, provided a sufficiently high order n of Taylor's formula is chosen (with an increase in n, factorial increases faster than a power). In particular, for the function $f(x) = \sin x$ and $x_0 = 0$ the following estimate of the remainder holds true:

$$|r_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}.$$

If we want to evaluate $\sin x$ for $x = \frac{\pi}{2}$ to within 0.01, then we choose n from the condition

$$\frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!} \leqslant 0.01,$$

which is satisfied for $n \geqslant 7$. Thus, the Taylor polynomial of the seventh order

$$P_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{6720} \approx \sin x$$

enables us to compute the function $f(x) = \sin x$ to within 0.01 on the entire interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Figure 107 shows how the

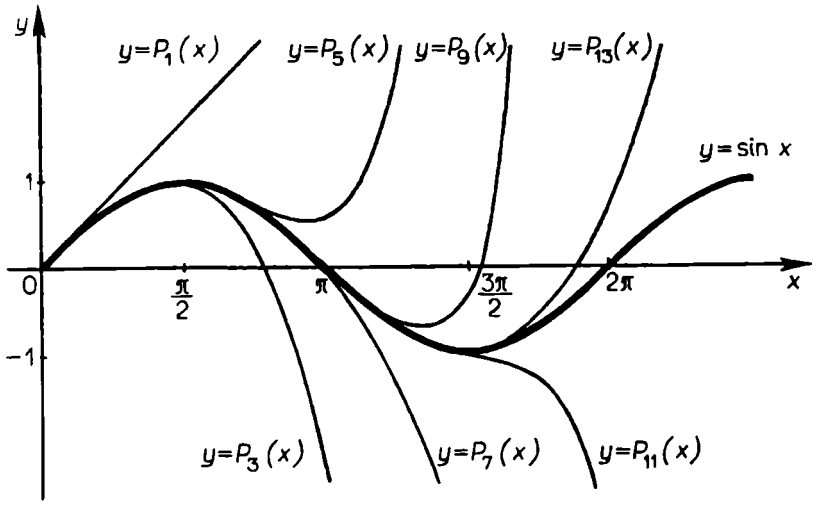

Fig. 107

"sphere of action" of Taylor's polynomials $P_n(x)$ broadens for $f(x) = \sin x$ with an increase in n (since the function $\sin x$ is odd, only "one-half" of the graph is shown for $x \geqslant 0$).

2. Taylor's Formulas for the Basic Elementary Functions. Let us apply Taylor's formula with the remainder in Lagrange's form to the basic elementary functions x^α, e^x, $\ln x$, $\sin x$, and $\cos x$ (Tay-

lor's formulas for the functions tan x, cot x, and for the inverse trigonometric functions look rather awkward and are used not so frequently in practice). For the functions e^x, sin x, and cos x it is convenient to choose $x_0 = 0$, whereas for the functions x^α and ln x it is more convenient to put $x_0 = 1$. To have in all the cases $x_0 = 0$, we shall consider the functions $(1 + x)^\alpha$ and ln $(1 + x)$ instead of the functions x^α and ln x. The particular case of Taylor's formula when $x_0 = 0$ is called Maclaurin's formula.

For the function $f(x) = e^x$ we have $f'(x) = e^x$, $f''(x) = e^x$, and, in general, $f^{(n)}(x) = e^x$. Therefore Taylor's formula for the function e^x at the point $x_0 = 0$ has the form:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n + \frac{e^c}{(n+1)!}x^{n+1}, \qquad (1)$$

where c lies between x and zero.

For the function $f(x) = \sin x$ we find $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, etc., so that for even $n = 2l$ we have $f^{(n)}(0) = 0$, while for odd $n = 2l + 1$ we have $f^{(n)}(0) = f^{(2l+1)}(0) = (-1)^l$. Finally, we have

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \frac{(-1)^{n+1}\cos c}{(2n+3)!}x^{2n+3}, \qquad (2)$$

where c lies between zero and x.

Similarly, for the function cos x we get:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + \frac{(-1)^{n+1}\cos c}{(2n+2)!}x^{2n+2}, \qquad (3)$$

where c lies between 0 and x.

For the function $f(x) = \ln(1 + x)$ we compute the derivatives:

$$f'(x) = \frac{1}{(1+x)}, \quad f''(x) = -\frac{1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

and, in general,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Therefore Taylor's formula for the function ln $(1 + x)$ at the point $x_0 = 0$ has the form:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots + \frac{(-1)^{n-1}}{n}x^n - \frac{(-1)^n}{(n+1)(1+c)^{n+1}}x^{n+1}, \qquad (4)$$

where c lies between x and zero.

For the function $f(x) = (1 + x)^\alpha$ we find $f'(x) = \alpha(1 + x)^{\alpha-1}$, $f''(x) = \alpha(\alpha - 1)(1 + x)^{\alpha-2}$, and, in general,

$$f^{(n)}(x) = \alpha(\alpha - 1) \ldots (\alpha - n + 1)(1 + x)^{\alpha-n},$$

so that

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!}(1+c)^{\alpha-n-1}\cdot x^{n+1}, \quad (5)$$

where c lies between x and zero. In the particular case $\alpha = -1$ we have:

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\frac{(-1)^{n+1}x^{n+1}}{(1+c)^{n+2}}.$$

Using formulas (1-5), we can obtain Taylor's formulas of a low order for composite functions.

Example. Expand the function $f(x) = e^{\sin x}$ by Taylor's formula at the point $x_0 = 0$ to the third order.

△ For the function e^y the following expansion is valid (see formula (1)):

$$e^y=1+y+\frac{y^2}{2}+\frac{y^3}{6}+\frac{e^c y^4}{4!},$$

where c lies between y and zero. Setting $y = \sin x$ in this formula we get:

$$e^{\sin x}=1+\sin x+\frac{(\sin x)^2}{2}+\frac{(\sin x)^3}{6}+\frac{e^c(\sin x)^4}{4!}.$$

From formula (2) we find:

$$\sin x=x-\frac{x^3}{6}+\frac{\cos\eta\cdot x^5}{5!},$$

where η lies between x and zero. Then

$$e^{\sin x}=1+x-\frac{x^3}{6}+\frac{\cos\eta\cdot x^5}{5!}+\frac{1}{2}\left(x-\frac{x^3}{6}+\frac{\cos\eta}{5!}x^5\right)^2 + \frac{1}{6}\left(x-\frac{x^3}{6}+\ldots\right)^3+\frac{e^c}{4!}\left(x-\frac{x^3}{6}+\ldots\right)^4 = 1+x-\frac{x^3}{6}+\frac{1}{2}x^2+\frac{1}{6}x^3+\varphi(x)x^4=1+x+\frac{x^2}{2}+\varphi(x)x^4,$$

where $\varphi(x)$ is some continuous function. ▲

3. Calculating Elementary Functions on a Computer. Consider, for example, Taylor's formula of the nth order of the function $f(x) = e^x$ for $x_0 = 0$:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\frac{e^c x^{n+1}}{(n+1)!},$$

where c lies between x and zero.

For the remainder the following estimate is valid:

$$|r_n| = \frac{e^c \cdot |x^{n+1}|}{(n+1)!} \leqslant \frac{e^{|x|} \cdot |x^{n+1}|}{(n+1)!}.$$

Hence it is obvious that the accuracy of the approximate formula

$$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}, \qquad (1)$$

obtained as a result of rejecting the remainder term depends both on the magnitude of the argument x and the order n of Taylor's polynomial. For sufficiently large $|x|$, to satisfy the inequality

$$|r_n| \leqslant \frac{e^{|x|} \cdot |x^{n+1}|}{(n+1)!} \leqslant \varepsilon,$$

ensuring the preassigned accuracy of computing ε $(0 < \varepsilon < 1)$, it is required to take a sufficiently large order n of Taylor's polynomial in formula (1). As a result, the "manual" computations of the function e^x by this formula become extremely difficult since a great volume of computations is involved. The application of an engineer's calculator performing arithmetic operations with a high speed does not change the situation in a cardinal way since the operator, as before, has to check the steps of computing process, namely, to press the calculator keys in order to realize all necessary computations, to record intermediate results, to watch the order of performing arithmetic operations, etc.

In contrast to man, an electronic computer readily copes with great amounts of computations.

In order to compute the function e^x by formula (1) on a computer, we have to write the computation program in one of programming languages. For this purpose, let us briefly consider the principal elements of the programming language BASIC as one of the most simple and widely used.

Variable quantities are written in this language with the help of capital letters of the Latin alphabet. Arithmetic operations are denoted by the symbols $+$, $-$, / (division), $*$ (multiplication), and $\uparrow$ (involution). The whole program is a sequence of numbered lines representing some operation or direction (instruction). The last line of the program contains the word "END".

We denote the argument x and the order n of Taylor's polynomial by the letters X and N, respectively, and shall introduce them as initial numerical data using as the first line of the program the input instruction INPUT ␣ X, N, the symbol ␣ being used here for a distinct designation of the spacing between the word INPUT and the quantities being introduced.

Sum (1) consisting of $(n + 1)$ terms is computed by assigning to the current (changing in the process of computation) sum S first

its initial value equal to unity, and then adding in turn x, $\frac{x^2}{2!}$, $\frac{x^3}{3!}$, etc. Let us explain the assigning instruction in BASIC by the example LET ␣ S = 1. With the help of this instruction, the sum S will be assigned its initial value equal to unity.

The values of terms A_k in sum (1) is conveniently computed in terms of their preceding values A_{k-1} (that is, recurrently), since for any $k \geqslant 1$

$$A_k = \frac{x^k}{k!} = \left(\frac{x}{k}\right) \cdot \frac{x^{k-1}}{(k-1)!} = \left(\frac{x}{k}\right) A_{k-1}, \quad A_0 = 1. \tag{2}$$

Equality (2) in BASIC looks at first sight as a paradox:

$$\text{LET} \; \text{␣} \; \text{A} = \text{X} * \text{A/K}, \tag{3}$$

since both sides of the assigning operator use the same letter A to denote different (preceding and consecutive) values of the addend of sum (1). Therefore let us explain the order of executing the assigning instruction: we first compute the expression to the right of the equality sign, and then the numerical value obtained is assigned to the variable quantity to the left of the equality sign. Thereby instruction (3) computes the new value of the term A using its current (that is, preceding) value according to formula (2). The initial value of A is equal to unity. We now write out the entire program for computing sum (1):

```
1 ␣ INPUT ␣ X, N
2 ␣ LET ␣ S = 1
3 ␣ LET ␣ A = 1
4 ␣ FOR ␣ K = 1 ␣ TO ␣ N
5 ␣ LET ␣ A = X * A/K
6 ␣ LET ␣ S = S + A
7 ␣ NEXT ␣ K
8 ␣ PRINT ␣ S
9 ␣ END
```

Here the instruction PRINT ␣ S is used to display the results of computation (of the sum S) on the television screen of a personal computer. The instruction of the cycle

$$\text{FOR} \; \text{␣} \; \text{K} = 1 \; \text{␣} \; \text{TO} \; \text{␣} \; \text{N}$$

means that all the subsequent instructions up to the limiting line NEXT ␣ K will cyclically repeat N times, the integer variable K keeping increasing by unity from its initial value K = 1 to the final value equal to N. Thereby the new value of the addend is computed in the fifth line of the program and is added to the current value

of the sum in the sixth line. Repeating (according to the program) these operations N times, the computer will calculate the desired sum S.

Figure 108 shows the graphs of the function e^x and Taylor's polynomials $P_n(x)$ calculated by the considered program on a personal

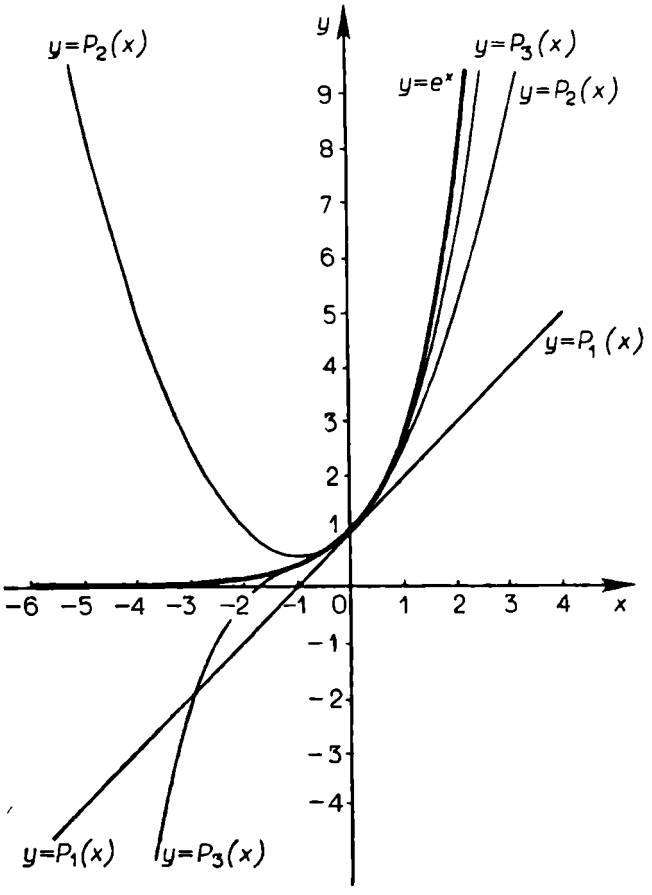

Fig. 108

computer. It is obviously seen how the "sphere of action" of Taylor's polynomials $P_n(x)$ broadens with an increase in n.

Exercises

1. Using Taylor's formula, expand at the point $x_0 = 0$ the following functions:

(a) ln (cos x) up to the fourth order;
(b) sin (sin x) up to the third order;
(c) tan x up to the fifth order.

2. What error has each of the given approximations:

(a) $\tan x \approx x + \frac{x^3}{3}$ on the interval $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$;

(b) $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ on the interval $[0, 1]$?

3. Write in BASIC the program of computation of Taylor's polynomial of the nth order for the function $\sin x$.

6.12. Approximate Computation of Equation Roots

1. The "Bracket" Method. We often fail to find an exact solution of the equation

$$f(x) = 0, \tag{1}$$

where $f(x)$ is a continuous function on a closed interval $[a, b]$. Therefore we are interested in approximate methods of finding the roots of equation (1). We shall assume that on the interval $[a, b]$ equation (1) has only one root not coinciding with the endpoints of the closed interval, and the function $f(x)$ attains values of opposite signs at the points $x=a$ and $x=b$. For definiteness, we set $f(a) < 0$ and $f(b) > 0$. In practice, we often find closed intervals possessing the indicated properties from graphical considerations. Figure 109 shows the graphs of the functions $y = \cos x$ and $y = x$, whence it is clear that the equation

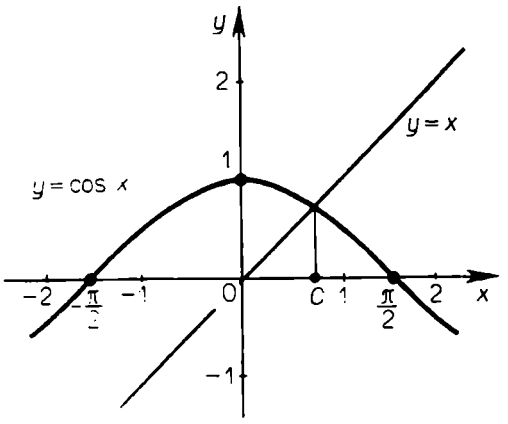

Fig. 109

$$f(x) = x - \cos x = 0 \tag{2}$$

has one root $x = c$ on the interval $\left[0, \frac{\pi}{2}\right]$ which does not coincide with its endpoints, and $f(0) = -1 < 0$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0$.

Henceforward, for brevity, any closed interval at whose endpoints a function $f(x)$ has values of opposite signs will be called a "bracket". Now, we are going to describe the method of approximately finding the root of the equation $f(x)=0$ which is called the "bracket" method. We divide the closed interval $[a, b]$ into two equal parts. Two cases are possible here: (1) the value of the function $f(x)$ at the midpoint of the interval $[a, b]$ is equal to zero (in this case, the desired root has been found: $c = \frac{a+b}{2}$); (2) the mentioned

value is not equal to zero. In the latter case, one of the two obtained halves of the interval $[a, b]$ is a "bracket". This half will be denoted by $[a_1, b_1]$. In order to obtain $f(a_1) < 0$, $f(b_1) > 0$, we choose

$$a_1 = a, \quad b_1 = \frac{a+b}{2} \quad \text{if} \quad f\left(\frac{a+b}{2}\right) > 0,$$

and

$$a_1 = \frac{a+b}{2}, \quad b_1 = b \quad \text{if} \quad f\left(\frac{a+b}{2}\right) < 0,$$

that is, the endpoints of a new "bracket" are determined depending upon the sign of the function at the midpoint.

Now, we deal with the closed interval $[a_1, b_1]$ whose length is half the length of the initial interval. We then proceed in the same way, that is, we bisect $[a_1, b_1]$ and so on. Continuing the described process, we shall have two possibilities: (1) either the above process will terminate owing to the fact that the value of the function at the midpoint of some of the subintervals will turn out to be equal to zero (in this case, the desired root is found); (2) or it will continue without bound, and we shall obtain a contracting system of intervals ("brackets"): $[a_1, b_1]$, $[a_2, b_2]$, . . ., $[a_n, b_n]$. The length of each subsequent bracket is one-half of the length of the preceding one, so that the length of the interval $[a_n, b_n]$ is equal to $\frac{b-a}{2^n}$ and can be very small for a sufficiently large n. Then, if we take the number a_n (the left endpoint of the nth bracket) for an approximate value of the root, that is, if we set $c \approx a_n$, we shall make a mistake not exceeding $\frac{b-a}{2^n}$. If we are given the accuracy ε of finding the root, then from the inequality $\frac{b-a}{2^n} < \varepsilon$ we can find the necessary number of divisions n:

$$n > \log_2\left(\frac{b-a}{\varepsilon}\right).$$

The described process of the approximate finding of the root of equation (1) consists in multiple repetition of computational operations of the same type which makes it convenient to be realized on a computer.

2. Programs in the BASIC Language. Prior to writing a computation program in BASIC, it is convenient to represent the computational algorithm described in the preceding subsection as an obvious diagram in which the letter A denotes the left-hand endpoint of the interval (its coordinate), B the right-hand endpoint of the interval, E the accuracy of finding the root, D the midpoint of the interval, and G the value of the function $f(x)$ at the point D (Fig. 110).

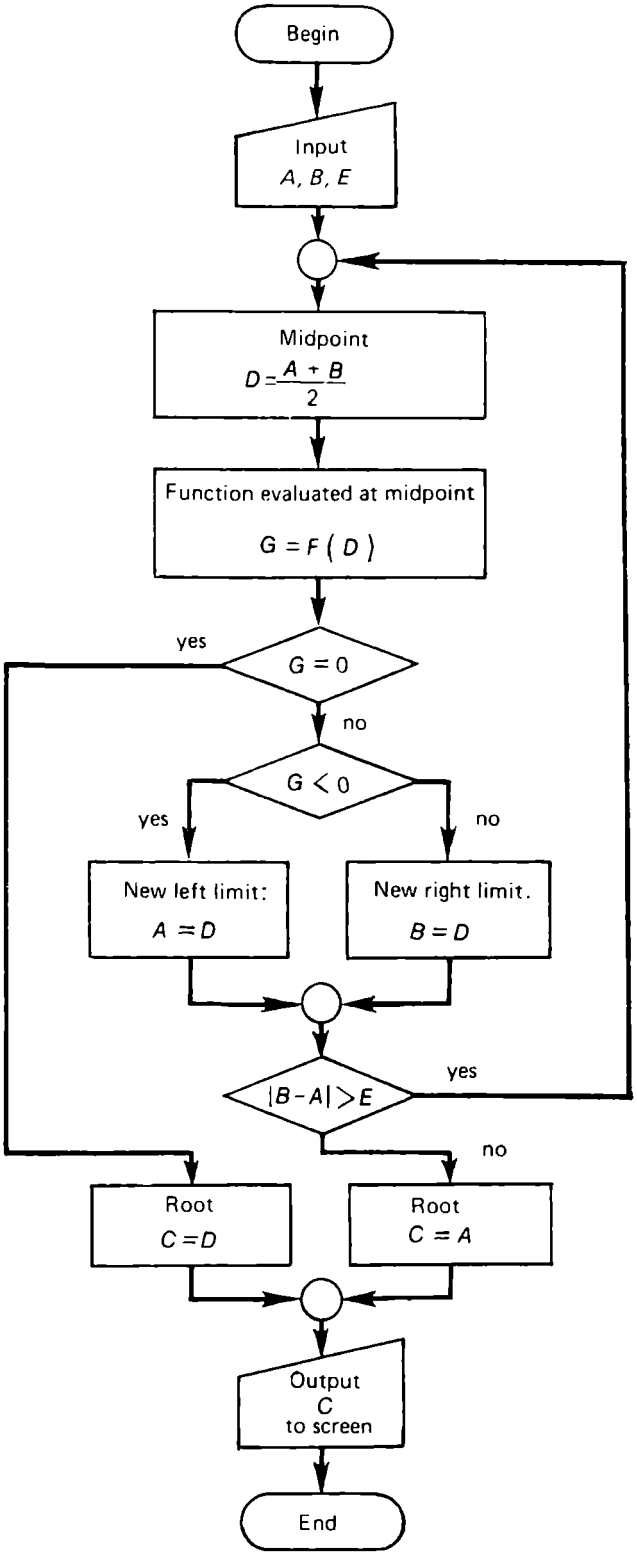
Begin
Input
A, B, E
Midpoint
D = (A + B)/2
Function evaluated at midpoint
G = F (D)
yes
G = 0
no
G < 0
yes
no
New left limit:
A = D
New right limit.
B = D
yes
|B - A| > E
no
Root
C = D
Root
C = A
Output
C
to screen
End

Fig. 110

Enclosed in a rhombus on algorithm diagrams is the condition depending on which the computational algorithm follows one or another course. For instance, if the condition "G = 0" is satisfied which means that the function vanishes at the midpoint of the interval, then computations come to an end. Otherwise, computations are continued. To realize such a branching of the computational algorithm, the BASIC language uses a conditional transfer instruction which can have, for instance, the form:

IF ␣ G = 0 ␣ THEN ␣ 570,

where the equality G = 0 is written as a condition. If this equality occurs, then the line with number 570 will be executed next in the program, and if the condition does not occur, then the line following the conditional transfer instruction is executed.

Now it is easy to write the program in BASIC for solving the equation $x - \cos x = 0$ corresponding to the algorithm diagram shown in Fig. 110:

```
 1 INPUT A, B, E
 2 LET D = (A + B)/2
 3 LET G = D — COS (D)
 4 IF G = 0 THEN 12
 5 IF G < 0 THEN 8
 6 LET B = D
 7 GOTO 9
 8 LET A = D
 9 IF ABS (B — A) > E THEN 2
10 LET C = A
11 GOTO 13
12 LET C = D
13 PRINT C
14 END
```

The unconditional transfer instruction GOTO ␣ TO ␣ 9 means the requirement to pass over to executing the instruction written in line 9. The cosine and absolute value are denoted as COS and ABS, respectively.

The initial data A = 0, B = $\frac{\pi}{2}$ and accuracy E are put in from the terminal (information input device) by already known instruction INPUT ␣ A, B, E. For $E = 10^{-1}$ we find C = 0.686875, for $E = 10^{-2}$, C = 0.7359375, and for $E = 10^{-3}$, we have C = 0.7390039. Thus, the bracket method enables us to find the root of an equation with any preassigned accuracy.

Exercises

Write the programs in BASIC for solving the following equations by the bracket method with an accuracy to $\varepsilon = 0.01$:

(a) $f(x) = \sin x - a = 0$, $x \in \left[0, \frac{\pi}{2}\right]$, where $0 < a < 1$;

(b) $f(x) = 3x - e^x = 0$, $x \in [0, 1]$;

(c) $f(x) = \tan x + x - 1 = 0$, $x \in [0, 1]$.

Chapter 7

INDEFINITE INTEGRALS

7.1. The Indefinite Integral and Its Properties

1. The Antiderivative and Indefinite Integral. As is known, in differential calculus we concentrated on solving the problem of finding the derivative or the differential of a given function. However, practice shows that, given a derivative or, which is the same, a differential, we have to find the function from which the derivative was taken, that is, to solve a problem inverse to differentiation.

Finding a function by its derivative or differential is studied in integral calculus. The function restored by its derivative or differential is called the antiderivative (or primitive).

Definition 1. A function $F(x)$ is an antiderivative (or a primitive function) of a function $f(x)$ in some interval if

$$F'(x) = f(x) \tag{1}$$

for any x from this interval.

The function $F(x) = x^4$ is an antiderivative of the function $f(x) = 4x^3$ since $F'(x) = 4x^3 = f(x)$ for any $x \in \boldsymbol{R}$. It is easy to see that the functions $F_1(x) = x^4 + 1$, $F_2(x) = x^4 - 2$, and, in general, $F_3(x) = x^4 + C$, where C is an arbitrary constant, are also antiderivatives of the function $f(x) = 4x^3$.

This example shows that a function having one antiderivative has infinitely many antiderivatives. We are going to show how to find all antiderivatives of a given function, knowing one of them.

Theorem 7.1.1. *If the function $F(x)$ is an antiderivative of the function $f(x)$ on some interval, then the set of all antiderivatives of the function $f(x)$ is given by the formula*

$$F(x) + C, \quad C \in \boldsymbol{R}. \tag{2}$$

□ It is easy to see that any function of the form $F(x) + C$, where C is a certain constant, is an antiderivative of $f(x)$. Indeed,

$$(F(x) + C)' = F'(x) = f(x).$$

Let us now prove that any antiderivative of $f(x)$ is representable in the form $F(x) + C$, $C \in \boldsymbol{R}$. Let $\Phi(x)$ be an antiderivative of $f(x)$, that is, $\Phi'(x) = f(x)$. We shall consider an auxiliary function

$$\varphi(x) = \Phi(x) - F(x)$$

and show that it is constant. Let x_1 and x_2 be two arbitrary points belonging to the interval under consideration and let, for definiteness, $x_1 < x_2$. According to Lagrange's theorem, there is a point $\xi \in (x_1, x_2)$ such that

$$\varphi(x_2) - \varphi(x_1) = \varphi'(\xi)(x_2 - x_1).$$

Since $\varphi'(x) = \Phi'(x) - F'(x) = f(x) - f(x) = 0$, and, in particular, $\varphi'(\xi) = 0$, we have $\varphi(x_2) = \varphi(x_1)$. Since x_1 and x_2 are arbitrary, we have: $\varphi(x) = C$, where C is a constant, that is

$$\Phi(x) = F(x) + C, \quad x \in (a, b). \quad \blacksquare$$

Definition 2. The collection of all antiderivatives of a given function $f(x)$ is called the indefinite integral of $f(x)$ and is denoted by the symbol

$$\int f(x)\,dx. \tag{3}$$

This symbol is read: "Integral of $f(x)$ with respect to x". The symbol $\int$ is called the integral sign, the function $f(x)$ the integrand, the expression $f(x)\,dx$ the element of integration, and x the variable of integration.

Note that if $F(x)$ is an antiderivative of the function $f(x)$, then, according to the definition, we may write:

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

Finding a function by its derivative (or by its differential) is called integration. Integration is the operation inverse to differentiation. The correctness of integration can always be checked by differentiating the result. For instance,

$$\int (2x - 7)\,dx = x^2 - 7x + C,$$

since

$$(x^2 - 7x + C)' = 2x - 7.$$

For an indefinite integral the following formulas are valid:

$$\left(\int f(x)\,dx\right)' = f(x),$$

$$d\int f(x)\,dx = f(x)\,dx,$$

$$\int f'(x)\,dx = f(x) + C,$$

$$\int df(x) = f(x) + C,$$

$$\int af(x)\,dx = a\int f(x)\,dx, \quad a \in \boldsymbol{R}, \quad a \neq 0,$$

$$\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx.$$

The correctness of these formulas is checked by differentiation.

2. Direct Integration. From the definition of the integral it follows that any formula for the derivative of some function, that is, a formula of the form $F'(x) = f(x)$ may be written in the form of the integral formula

$$\int f(x)\,dx = F(x) + C.$$

Thus, using the table of derivatives of simplest elementary functions (see Table 1 of 6.4), we compile a table of simplest integrals.

Table 1

1. $\int 0\,dx = C$	7. $\int \frac{dx}{\cos^2 x} = \tan x + C$
2. $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \ \alpha \neq -1$	8. $\int \frac{dx}{\sin^2 x} = -\cot x + C$
3. $\int a^x dx = \frac{a^x}{\ln a} + C$	9. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
4. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	10. $\int \frac{dx}{1+x^2} = \arctan x + C$
5. $\int \cos x\,dx = \sin x + C$	
6. $\int \sin x\,dx = -\cos x + C$	

The integrals listed in Table 1 will be referred to as tabular (or standard) integrals.

Direct integration is such a method of computing integrals in which they are reduced to the tabular formulas by applying to them the basic properties of indefinite integrals.

Example 1. Evaluate the integral $\int x\sqrt{x}\,dx$.

△ Since $x\sqrt{x} = x^{\frac{3}{2}}$, we have $\int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{2}{5}x^{\frac{5}{2}} + C$. ▲

Example 2. Evaluate the integral $\int (\sqrt[5]{x})^{-9}\,dx$.

△ Since $(\sqrt[5]{x})^{-9} = x^{-\frac{9}{5}}$, we have

$$\int (\sqrt[5]{x})^{-9}\,dx = \int x^{-\frac{9}{5}}\,dx = \frac{x^{-\frac{9}{5}+1}}{-\frac{9}{5}+1} + C = -\frac{5}{4}\,x^{-\frac{4}{5}} + C. \blacktriangle$$

Example 3. Evaluate the integral $\int (t+1)^2\,dt$.

△ Since $(t^2+1)^2 = t^4+2t^2+1$, we get

$$\int (t^2+1)^2\,dt = \int (t^4+2t^2+1)\,dt = \int t^4\,dt + 2\int t^2\,dt + \int dt$$
$$= \frac{1}{5}\,t^5 + \frac{2}{3}\,t^3 + t + C. \blacktriangle$$

Example 4. Evaluate the integral $\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx$.

△ Since $\sin\frac{x}{2}\cos\frac{x}{2} = \frac{1}{2}\sin x$, we have

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx = \frac{1}{2}\int \sin x\,dx = -\frac{1}{2}\cos x + C. \blacktriangle$$

Exercises

1. Evaluate the following integrals:

(a) $\int x^7\,dx$; (b) $\int x^{1988}\,dx$; (c) $\int x^{-99}\,dx$; (d) $\int \frac{t^9\,dt}{\sqrt{t}}$;

(e) $\int (\sqrt{x})^{11}\,dx$; (f) $\int e^x\,dx$; (g) $\int 7^x\,dx$; (h) $\int t^{\alpha-1}\,dt$;

(i) $\int t^{\beta+6}\,dt$; (j) $\int t^{\beta+6}\,d\beta$.

2. Evaluate the following integrals:

(a) $\int \left(x^2+2x+\frac{1}{x}\right)dx$; (b) $\int \frac{10x^6+3}{x^4}\,dx$; (c) $\int \frac{x-2}{x^3}\,dx$;

(d) $\int \frac{(x^2+1)}{x^3}dx$; (e) $\int \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt[4]{x^3}}\right)dx$; (f) $\int \frac{(\sqrt{x}-1)^3}{x}\,dx$;

(g) $\int \frac{x-1}{\sqrt[3]{x^2}}\,dx$.

3. Evaluate the following integrals:

(a) $\int e^x \left(1-\frac{e^{-x}}{x^2}\right) dx$; (b) $\int a^x \left(1+\frac{a^{-x}}{\sqrt{x^3}}\right) dx$;

(c) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$; (d) $\int \left(\frac{2}{1+x^2}-\frac{3}{\sqrt{1-x^2}}\right) dx$;

(e) $\int \left(\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}\right) dx$; (f) $\int \left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2 dx$;

(g) $\int e^x \left(1+\frac{e^{-x}}{\cos^2 x}\right) dx$; (h) $\int \left(1-2\sin^2\frac{x}{2}\right) dx$;

(i) $\int \frac{dx}{1-\sin^2 x}$; (j) $\int \frac{dt}{1-\cos^2 t}$;

(k) $\int \frac{du}{\sqrt{1-u^2}}$; (l) $\int \frac{du}{\sqrt{2u+2-(u+1)^2}}$;

(m) $\int \frac{dx}{(x+1)^2-2x}$.

7.2. Methods of Integration

1. Integration by Change of Variable (by Substitution). The integral $\int f(x)\,dx$ can be often simplified if, instead of x, we introduce a new variable of integration t by setting

$$x = \varphi(t).$$

Then $f(x) = f(\varphi(t))$, $dx = \varphi'(t)\,dt$ and

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt\Big|_{t=\varphi^{-1}(x)}, \qquad (1)$$

where in the final result we return to the variable x using the formula $t = \varphi^{-1}(x)$. Formula (1) is called the formula for changing variable in indefinite integrals.

Example 1. Evaluate the integral $\int \frac{1}{x^2} e^{\frac{1}{x}} dx$.

△ We set $x=\frac{1}{t}$. Then $dx=-\frac{dt}{t^2}$, and, consequently,

$$\int \frac{1}{x^2} e^{\frac{1}{x}} dx = \int t^2 e^t \left(-\frac{dt}{t^2}\right) = -\int e^t\,dt = -e^t + C = -e^{\frac{1}{x}} + C. \ ▲$$

Example 2. Evaluate the integral $\int \sin 3x\,dx$.

△ We set $x=\frac{t}{3}$. Then $\int \sin 3x\,dx = \int \sin t\,d\left(\frac{t}{3}\right) = \frac{1}{3}\int \sin t\,dt =$

$$-\frac{1}{3}\cos t + C = -\frac{1}{3}\cos 3x + C. \ ▲$$

Sometimes, instead of changing the variable $x = \varphi(t)$ it is better to carry out the change of variable of the form $t = \psi(x)$.

Example 3. Evaluate the integral $\int \frac{dx}{1+\sqrt{1+x}}$.

△ We introduce a new variable: $t=1+\sqrt{1+x}$. Then $x=(t-1)^2-1$, $dx=x'(t)\,dt=2(t-1)\,dt$, and, consequently,

$$\int \frac{dx}{1+\sqrt{1+x}}=\int \frac{2(t-1)\,dt}{t}=2\int dt-2\int \frac{dt}{t}=2t-2\ln|t|+C_1$$

$$=2(1+\sqrt{1+x})-2\ln|1+\sqrt{1+x}|+C_1$$

$$=2\sqrt{1+x}-2\ln|1+\sqrt{1+x}|+C. \blacktriangle$$

Example 4. Evaluate the integral $\int \frac{dx}{x+5}$.

△ Since $dx=d(x+5)$, introducing a new variable $t=x+5$, we find:

$$\int \frac{dx}{x+5}=\int \frac{d(x+5)}{x+5}=\int \frac{dt}{t}=\ln|t|+C=\ln|x+5|+C. \blacktriangle$$

Example 5. Evaluate the integral $I=\int x\sqrt{1-x^2}\,dx$.

△ It is easy to see that $x\,dx=\frac{1}{2}dx^2$. Then

$$I=\int x\sqrt{1-x^2}\,dx=\frac{1}{2}\int \sqrt{1-x^2}\,dx^2=-\frac{1}{2}\int \sqrt{1-x^2}\,d(1-x^2).$$

Introducing a new variable $t=1-x^2$, we find:

$$I=-\frac{1}{2}\int \sqrt{t}\,dt=-\frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=-\frac{1}{3}t^{\frac{3}{2}}+C$$

$$=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C. \blacktriangle$$

Example 6. Evaluate the integral $\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx$.

△ We set $t=\arcsin x$. Then $dt=\frac{dx}{\sqrt{1-x^2}}$, and, consequently,

$$\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx=\int t^2\,dt=\frac{t^3}{3}+C=\frac{1}{3}(\arcsin x)^3+C. \blacktriangle$$

Example 7. Evaluate the integral $\int \frac{dx}{\sqrt{1-9x^2}}$.

△ We set $x=\frac{t}{3}$. Then

$$\int \frac{dx}{\sqrt{1-9x^2}}=\frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}}=\frac{1}{3}\arcsin t+C=\frac{1}{3}\arcsin 3x+C. \blacktriangle$$

Example 8. Evaluate the integral $\int \frac{dx}{\sqrt{a^2-x^2}}$.

△ We set $x=at$. Then $\int \frac{dx}{\sqrt{a^2-x^2}}=\int \frac{dt}{\sqrt{1-t^2}}=\arcsin t+C=$ $\arcsin \frac{x}{a}+C. \blacktriangle$

Example 9. Evaluate the integral $\int \frac{dx}{25+4x^2}$.

△ We set $x=\frac{5t}{2}$. Then $\int \frac{dx}{25+4x^2}=\frac{1}{10}\int \frac{dt}{1+t^2}=\frac{1}{10}\arctan t+$ $C=\frac{1}{10}\arctan \frac{2x}{5}+C. \blacktriangle$

Example 10. Evaluate the integral $\int \tan x\,dx$.

△ Let us make use of the equality $d\cos x = -\sin x\,dx$ and introduce a new variable of integration: $t = \cos x$:

$$\int \tan x\,dx=\int \frac{\sin x\,dx}{\cos x}=-\int \frac{d\cos x}{\cos x}=-\int \frac{dt}{t}$$
$$=-\ln|\cos x|+C. \blacktriangle$$

2. Integration by Parts. According to the known rule for differentiation of a product, we have

$$d(uv) = v\,du + u\,dv,$$

whence

$$u\,dv = d(uv) - v\,du.$$

If the derivatives (or, which is the same, differentials) of two functions are equal to each other, then their indefinite integrals coincide. Therefore

$$\int u\,dv=\int d(uv)-\int v\,du,$$

and, consequently,

$$\int u\,dv=uv-\int v\,du. \tag{1}$$

This formula is called the formula of integration by parts. It is successfully used in the cases when it is easier to compute the integral $\int v\,du$ than the initial integral $\int u\,dv$.

Example 1. Find the integral $\int \arctan x\,dx$.

△ Setting $u = \arctan x$, $dv = dx$ and applying the formula for integration by parts, we get

$$\int \arctan x\,dx = (\arctan x)\cdot x - \int x\,d\arctan x = x\arctan x - \int \frac{x\,dx}{1+x^2}$$
$$= x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = x\arctan x - \frac{1}{2}\int \frac{dz}{z}$$
$$= x\arctan x - \frac{\ln|z|}{2} + C = x\arctan x$$
$$- \frac{1}{2}\ln(1+x^2) + C.$$

We have omitted the modulus sign since $(1 + x^2) > 0$ for any $x \in R$. ▲

Example 2. Find the integral $\int x\arctan x\,dx$.

△ We set $u = \arctan x$, $dv = x\,dx = \frac{1}{2}\,dx^2$ and apply formula (1):

$$\int x\arctan x\,dx = \frac{1}{2}\int \arctan x\,dx^2 = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int x^2 d\arctan x$$
$$= \frac{x^2\arctan x}{2} - \frac{1}{2}\int \frac{x^2\,dx}{1+x^2}$$
$$= \frac{x^2\arctan x}{2} - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$
$$= \frac{x^2\arctan x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$$
$$= \frac{x^2\arctan x}{2} - \frac{x}{2} + \frac{1}{2}\arctan x + C. \ ▲$$

Example 3. Find the integral $\int xe^x\,dx$.

△ Since $e^x\,dx = de^x$, we have

$$\int xe^x\,dx = \int x\,de^x.$$

Applying formula (1), we find:

$$\int x\,de^x = xe^x - \int e^x\,dx = xe^x - e^x + C. \ ▲$$

Example 4. Compute the integral $\int \ln x\,dx$.

△ Setting $u = \ln x$, $dv = dx$ and applying the formula for integration by parts, we get

$$\int \ln x\,dx = (\ln x)\cdot x - \int x\,d(\ln x) = x\ln x - \int \frac{x\,dx}{x} = x\ln x - x + C. \ ▲$$

Example 5. Compute the integral $\int (\arcsin x)^2\,dx$.

△ Integrating by parts, we find:

$$\int (\arcsin x)^2\,dx = x\,(\arcsin x)^2 - \int \frac{2x \arcsin x\,dx}{\sqrt{1-x^2}}.$$

To compute the last integral, we apply once again the method of integration by parts. Setting $u = \arcsin x$, we get

$$dv = \frac{2x\,dx}{\sqrt{1-x^2}} = \frac{dx^2}{\sqrt{1-x^2}} = -\frac{d\,(1-x^2)}{\sqrt{1-x^2}} = -2d\sqrt{1-x^2}.$$

Then

$$\begin{aligned}\int \frac{2x \arcsin x\,dx}{\sqrt{1-x^2}} &= -2\int \arcsin x\,d\sqrt{1-x^2}\\ &= -2\sqrt{1-x^2}\cdot\arcsin x + 2\int \frac{\sqrt{1-x^2}\,dx}{\sqrt{1-x^2}}\\ &= -2\sqrt{1-x^2}\arcsin x + 2x + C.\end{aligned}$$

Finally,

$$\int (\arcsin x)^2\,dx = x\,(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C. \ ▲$$

Exercises

1. Compute the following integrals by change of variable:

(a) $\int e^{-3x}\,dx$; (b) $\int \sin(a-bx)\,dx$; (c) $\int \sqrt{4x-1}\,dx$;

(d) $\int \frac{dx}{\sqrt{17-9x}}$; (e) $\int \frac{e^{2x}\,dx}{1-3e^{2x}}$; (f) $\int \cot x\,dx$;

(g) $\int \frac{\sin x\,dx}{1+3\cos x}$; (h) $\int \frac{\cos x\,dx}{1+2\sin x}$; (i) $\int \frac{dx}{x\,(1+\ln x)}$;

(j) $\int e^{\sin x}\cos x\,dx$; (k) $\int e^{x^3}x^2\,dx$; (l) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$;

(m) $\int \sqrt{x^3-8x^2}\,dx$; (n) $\int \frac{x\,dx}{\sqrt{1-x^2}}$; (o) $\int \frac{\sin x\,dx}{\sqrt{1+2\cos x}}$;

(p) $\int \sin^3 x\cos x\,dx$; (q) $\int \frac{x^3\,dx}{\sqrt{1-x^8}}$; (r) $\int \frac{1-2\cos x}{\sin^2 x}\,dx$.

2. Compute the following integrals by integration by parts:

(a) $\int x3^x\,dx$; (b) $\int x^n \ln x\,dx$; (c) $\int \arccos x\,dx$;

(d) $\int \arctan \sqrt{x}\,dx$; (e) $\int \frac{\arcsin x\,dx}{\sqrt{x+1}}$; (f) $\int \frac{\log_2 x}{x^3}\,dx$;

(g) $\int \frac{x \arctan x\,dx}{\sqrt{1+x^2}}$; (h) $\int \frac{\arcsin \sqrt{x}}{\sqrt{1-x}}\,dx$; (i) $\int \ln (x^2+1)\,dx$;

(j) $\int x^2 \sin x\,dx$.

7.3. Some Types of Integrals

1. Integrals of Rational Functions. Consider the rational function

$$R(x) = \frac{P_m(x)}{Q_n(x)}, \tag{1}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of degree m and n, respectively. Two cases are possible: (1) $m < n$, that is, a rational function is given in the form of a proper rational fraction; (2) $m \geqslant n$, that is, a rational function is given in the form of an improper rational fraction. In the last case, the numerator $P_m(x)$ is divided by the denominator $Q_n(x)$, and the result is obtained in the form of a sum of a polynomial and a proper rational fraction (which is the remainder after the division of $P_m(x)$ by $Q_n(x)$). Consequently, the soluton of the problem on integrating a rational function is reduced to the solution of the problem on finding the integral of a proper rational fraction. Let us consider some cases of finding integrals of proper rational fractions.

We shall begin with the simplest case $n = 1$, that is, when

$$Q_n(x) = Q_1(x) = a_0x + a_1.$$

If $m \geqslant 1$, then rational function (1) can be represented in the form

$$R(x) = P_{m-1}(x) + \frac{b_0}{a_0x+a_1},$$

where $P_{m-1}(x)$ is a polynomial of the $(m-1)$st degree. Computing the integral

$$\int \frac{b_0\,dx}{a_0x+a_1} = \frac{b_0}{a_0}\int \frac{d(a_0x+a_1)}{a_0x+a_1} = \frac{b_0}{a_0}\int \frac{du}{u}$$

$$= \frac{b_0}{a_0}\ln|u| + C = \frac{b_0}{a_0}\ln|a_0x+a_1| + C,$$

we finally find:

$$\int R(x)\,dx = \int P_{m-1}(x)\,dx + \frac{b_0}{a_0}\ln|a_0x+a_1|.$$

Example 1. Compute the integral $\int \frac{x^4-7x+11}{x-2}\,dx$.

△ Dividing the polynomial $x^4-7x+11$ by the polynomial $(x-2)$, we find:

$$\frac{x^4-7x+11}{x-2}=x^3+2x^2+4x+1+\frac{13}{x-2}.$$

Integrating, we get:

$$\int \frac{x^4-7x+11}{x-2}\,dx=\int x^3\,dx+2\int x^2\,dx+4\int x\,dx+\int dx+13\int \frac{dx}{x-2}$$
$$=\frac{x^4}{4}+\frac{2}{3}x^3+2x^2+x+13\ln|x-2|+C. \blacktriangle$$

Example 2. Compute the integral $\int \frac{x^3}{x+1}\,dx$.

△ Dividing the polynomial x^3 by the polynomial $(x+1)$, we find:

$$\frac{x^3}{x+1}=x^2-x+1-\frac{1}{x-1},$$

whence

$$\int \frac{x^3}{x+1}\,dx=\int x^2\,dx-\int x\,dx+\int dx-\int \frac{dx}{x+1}$$
$$=\frac{x^3}{3}-\frac{x^2}{2}+x-\ln|x+1|+C. \blacktriangle$$

Consider the case $n=2$, that is, when

$$Q_n(x)=Q_2(x)=a_0x^2+a_1x+a_2.$$

If $m\geqslant 2$, then rational function (1) can be represented in the form

$$R(x)=P_{m-2}(x)+\frac{b_0x+b_1}{a_0x^2+a_1x+a^2},$$

where $P_{m-2}(x)$ is a polynomial of the $(m-2)$nd degree. Computing the integral

$$\int \frac{b_0\,x\,dx}{a_0x^2+a_1x+a_2}=\frac{b_0}{2a_0}\int \frac{(2a_0x+a_1-a_1)\,dx}{a_0x^2+a_1x+a_2}$$
$$=\frac{b_0}{2a_0}\left[\int \frac{d(a_0x^2+a_1x+a_2)}{a_0x^2+a_1x+a_2}-a_1\int \frac{dx}{a_0x^2+a_1x+a_1}\right]$$
$$=\frac{b_0}{2a_0}\ln|a_0x^2+a_1x+a_2|-\frac{a_1b_0}{2a_0}\int \frac{dx}{a_0x^2+a_1x+a_2},$$

we find

$$\int R(x)\,dx=\int P_{m-2}(x)\,dx+\frac{b_0}{2a_0}\ln|a_0x^2+a_1x+a|$$
$$+\left(b_1-\frac{a_1b_0}{2a_0}\right)\int \frac{dx}{a_0x^2+a_1x+a_2}.$$

Example 3. Compute the integral $\int \frac{dx}{x^2+8x+7}$.

△ We single out a perfect square and factorize: $x^2 + 8x + 7 = (x^2 + 2x \cdot 4 + 16) - 16 + 7 = (x + 4)^2 - 9 = (x + 1)(x + 7)$. Expanding the fraction into a sum of partial fractions with unknown coefficients A and B, we get

$$\frac{1}{(x+1)(x+7)} = \frac{A}{x+1} + \frac{B}{x+7} = \frac{A(x+7)+B(x+1)}{(x+1)(x+7)} = \frac{(A+B)x+(7A+B)}{(x+1)(x+7)},$$

whence

$$1 = (A+B)x + (7A+B).$$

Equating the coefficients of equal powers of x in this identity we set up a system of equations

$$\begin{cases} A+B=0, \\ 7A+B=1. \end{cases}$$

Solving the system, we find $A = \frac{1}{6}$, $B = -\frac{1}{6}$. Consequently,

$$\int \frac{dx}{x^2+8x+7} = \int \frac{dx}{(x+1)(x+7)} = \frac{1}{6}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{dx}{x+7} = \frac{1}{6}\ln|x+1| - \frac{1}{6}\ln|x+7| + C = \frac{1}{6}\ln\left|\frac{x+1}{x+7}\right| + C. \blacktriangle$$

Example 4. Compute the integral $\int \frac{dx}{x^2+2x+5}$.

△ We separate a perfect square: $x^2+2x+5 = (x^2+2x+1) - 1 + 5 = (x+1)^2 + 4$ and compute the integral:

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\int \frac{d\left(\frac{x+1}{2}\right)}{1+\left(\frac{x+1}{2}\right)^2} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C. \blacktriangle$$

Example 5. Compute the integral $\int \frac{(x^4-3)\,dx}{x^2+2x+1}$.

△ Dividing the polynomial x^4-3 by the polynomial x^2+2x+1, we get:

$$\frac{x^4-3}{x^2+2x+1} = x^2 - 2x + 3 - \frac{4x+6}{x^2+2x+1}.$$

Integrating, we get:

$$\int \frac{(x^4-3)\,dx}{x^2+2x+1} = \int x^2\,dx - 2\int x\,dx + 3\int dx - 2\int \frac{2x+2+1}{x^2+2x+1}\,dx$$

$$= \frac{x^3}{3} - x^2 + 3x - 2\int \frac{d(x^2+2x+1)}{x^2+2x+1} - 2\int \frac{dx}{x^2+2x+1}$$

$$= \frac{x^3}{3} - x^2 + 3x - 2\ln(x+1)^2 - 2\int \frac{d(x+1)}{(x+1)^2}$$

$$= \frac{x^3}{3} - x^2 + 3x - 4\ln|x+1| + \frac{2}{x+1} + C. \blacktriangle$$

Further, consider the case $n \geqslant 3$. Suppose that the denominator of the rational fraction (1) is factorable (or already factorized) into a product of polynomials of degree not higher than two. In this case, a proper rational fraction ($m < n$) can be expanded into a sum of partial fractions whose integrals are found in the way considered above. Without dwelling on the solution of this problem in the general form, let us consider a number of examples.

Example 6. Compute the integral $\int \frac{dx}{x^3-1}$.

△ The integrand is a proper rational fraction whose denominator is readily factorized into a product: $x^3 - 1 = (x - 1)(x^2 + x + 1)$. We represent the fraction in the form of a sum:

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$

that is,

$$\frac{1}{x^3-1} = \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)},$$

or

$$\frac{1}{x^3-1} = \frac{(A+B)x^2+(A-B+C)x+(A-C)}{x^3-1},$$

whence

$$1 = (A + B)x^2 + (A - B + C)x + (A - C).$$

Equating the coefficients of equal powers of x in both sides of the identity, we form a system of equations

$$\begin{cases} A+B=0, \\ A-B+C=0, \\ A-C=1. \end{cases}$$

Solving the system, we find $A=\frac{1}{3}$, $B=-\frac{1}{3}$, $C=-\frac{2}{3}$. Consequently,

$$\int \frac{dx}{x^3-1}=\frac{1}{3}\int \frac{dx}{x-1}-\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\int \frac{d(x^2+x+1)}{x^2+x+1}$$

$$-\frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|x^2+x+1|-\frac{1}{\sqrt{3}}\int \frac{d\left(\frac{2x+1}{\sqrt{3}}\right)}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)$$

$$-\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C. \blacktriangle$$

Example 7. Compute the integral $\int \frac{(x-1)\,dx}{x^2(x-2)(x+1)^2}$.

△ The integrand is a proper rational fraction. We find the expansion of the integrand into partial fractions:

$$\frac{x-1}{x^2(x-2)(x+1)^2}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-2}+\frac{D}{(x+1)^2}+\frac{E}{x+1}.$$

Let us now reduce the right-hand side to a common denominator and equate the numerators: $x-1=A(x-2)(x+1)^2+Bx(x-2)(x+1)^2+Cx^2(x+1)^2+Dx^2(x-2)+Ex^2(x+1)(x-2)$.

Equating the coefficients of equal powers of x in both sides of the identity, we get a system of simultaneous equations

$$\begin{cases} B+C+E=0, \\ A+2C+D-E=0, \\ -3B+C-2D-2E=0, \\ -3A-2B=1, \\ -2A=-1. \end{cases}$$

Solving the system, we get $A=\frac{1}{2}$, $B=-\frac{5}{4}$, $C=\frac{1}{36}$, $D=\frac{2}{3}$, $E=\frac{11}{9}$. Thus,

$$\int \frac{(x-1)\,dx}{x^2(x-2)(x+1)^2}$$
$$=\frac{1}{2}\int \frac{dx}{x^2}-\frac{5}{4}\int \frac{dx}{x}+\frac{1}{36}\int \frac{dx}{x-2}+\frac{2}{3}\int \frac{dx}{(x+1)^2}+\frac{11}{9}\int \frac{dx}{x+1}$$
$$=-\frac{1}{2x}-\frac{5}{4}\ln|x|+\frac{1}{36}\ln|x-2|-\frac{2}{3(x+1)}+\frac{11}{9}\ln|x+1|+C. \blacktriangle$$

Example 8. Compute the integral $\int \frac{x-15}{x^3-2x^2+5x}\,dx$.

△ The integrand is a proper rational fraction whose denominator is readily factorized:

$$\frac{x-15}{x^3-2x^2+5x}=\frac{x-15}{x(x^2-2x+5)}=\frac{A}{x}+\frac{Bx+C}{x^2-2x+5}.$$

We reduce the right-hand side to a common denominator and equate the numerators:

$$x-15=(A+B)x^2+(-2A+C)x+5A.$$

Equating the coefficients of equal powers of x in both sides of the identity, we get the system of equations

$$\begin{cases} A+B=0, \\ -2A+C=1, \\ 5A=-15. \end{cases}$$

Solving the system, we get $A=-3$, $B=3$, $C=-5$. Thus,

$$\int \frac{x-15}{x^3-2x^2+5x}\,dx=-3\int \frac{dx}{x}+\int \frac{3x-5}{x^2-2x+5}\,dx$$
$$=-3\ln|x|+\frac{3}{2}\int \frac{(2x-2)+2-\frac{10}{3}}{x^2-2x+5}\,dx$$
$$=-3\ln|x|+\frac{3}{2}\int \frac{d(x^2-2x+5)}{x^2-2x+5}$$
$$-2\int \frac{d(x-1)}{(x-1)^2+4}=-3\ln|x|$$
$$+\frac{3}{2}\ln(x^2-2x+5)-\arctan\left(\frac{x-1}{2}\right)+C,$$

where $|x^2-2x+5|=x^2-2x+5$, since $x^2-2x+5=(x-1)^2+4>0$ for any $x\in R$. ▲

2. Integrals of Irrational Functions. When integrating irrational functions, we use changes of variables which enable us to reduce the

integration of irrational functions to integration of rational functions. For instance, to find integrals of the form

$$\int R(x, \sqrt{x}, \ldots, \sqrt[k]{x})\, dx,$$

where R is a rational function with respect to x and various fractional powers of x, we use the change of variable of the form $x = t^p$ (where p is the least common denominator of all the exponents of the power x). Consider a number of examples.

Example 1. Compute the integral $\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}$.

△ We set $x = t^6$. Then $\sqrt[3]{x} = t^2$, $\sqrt{x} = t^3$, $dx = 6t^5\, dt$. Therefore

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} = \int \frac{6t^5\, dt}{t^2+t^3} = 6\int \frac{t^3\, dt}{1+t}.$$

As a result we have obtained the integral of an improper rational fraction:

$$\begin{aligned} 6\int \frac{t^3\, dt}{1+t} &= 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right) dt \\ &= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C \\ &= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1+\sqrt[6]{x}) + C. \ \blacktriangle \end{aligned}$$

Example 2. Compute the integral $\int \frac{\sqrt{x+4}}{x}\, dx$.

△ We introduce a new variable: $t = \sqrt{x+4}$. Then $x = t^2 - 4$, $dx = 2t\, dt$, and, consequently,

$$\begin{aligned} \int \frac{\sqrt{x+4}}{x}\, dx &= \int \frac{2t^2\, dt}{t^2-4} = 2\int \left(1 + \frac{4}{t^2-4}\right) dt \\ &= 2t + 8\int \frac{dt}{(t-2)(t+2)}. \end{aligned}$$

Expanding the integrand into partial fractions, we get:

$$\frac{1}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{(A+B)t + (2A-2B)}{(t-2)(t+2)},$$

whence $A = \frac{1}{4}$, $B = -\frac{1}{4}$. Consequently, $\int \frac{\sqrt{x+4}}{x}\, dx = 2t + 2\int \frac{dt}{t-2} - 2\int \frac{dt}{t-2} = 2t + 2\ln|t-2| - 2\ln|t+2| + C = 2t + 2\ln\left|\frac{t-2}{t+2}\right| + C = 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$. ▲

Consider now a couple of examples in which so-called "trigonometric substitutions" are used.

Example 3. Compute the integral $\int \sqrt{1-x^2}\,dx$.

△ We set $x=\sin t$. Then $\sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=\cos t$, $dx=\cos t\,dt$. Therefore

$$\int \sqrt{1-x^2}\,dx=\int \cos^2 t\,dt=\int \frac{1+\cos 2t}{2}\,dt=\frac{t}{2}+\frac{\sin 2t}{4}+C$$

$$=\frac{1}{2}\arcsin x+\frac{1}{2}\,x\sqrt{1-x^2}+C,$$

since $\sin 2t=2\sin t\cos t=2\sin t\sqrt{1-\sin^2 t}=2x\sqrt{1-x^2}$. ▲

Example 4. Compute the integral $\int \sqrt{1+x^2}\,dx$.

△ We set $x=\tan t$. Then $\sqrt{1+x^2}=\sqrt{1+\tan^2 t}=\frac{1}{\cos t}$, $dx=\frac{dt}{\cos^2 t}$. Therefore

$$\int \sqrt{1+x^2}\,dx=\int \frac{dt}{\cos^3 t}=\int \frac{\cos t\,dt}{\cos^4 t}=\int \frac{d\sin t}{(1-\sin^2 t)^2}$$

$$=\int \frac{dz}{(1-z^2)^2}=\int \frac{dz}{(1-z)^2(1+z)^2},$$

where we, by setting $z=\sin t$, have come to the integral of a rational fraction. Let us expand the integrand into partial fractions:

$$\frac{1}{(1-z)^2(1+z)^2}=\frac{A}{(1-z)^2}+\frac{B}{1-z}+\frac{C}{(1+z)^2}+\frac{D}{1+z}.$$

Reducing the right-hand side to the common denominator, we equate the numerators:

$$1=(D-B)z^3+(A-B+C-D)z^2+(2A+B-2C-D)z$$
$$+(A+B+C+D).$$

Equating the coefficients of equal powers on both sides of the identity, we set up the system of equations

$$\begin{cases} D-B=0, \\ A-B+C-D=0, \\ 2A+B-2C-D=0, \\ A+B+C+D=1. \end{cases}$$

Solving the system, we find $A=B=C=D=\frac{1}{4}$. Consequently,

$$\int \sqrt{1+x^2}\,dx=\int \frac{dz}{(1-z)^2(1+z)^2}=\frac{1}{4}\int \frac{dz}{(1-z)^2}+\frac{1}{4}\int \frac{dz}{1-z}$$
$$+\frac{1}{4}\int \frac{dz}{(1+z)^2}+\frac{1}{4}\int \frac{dz}{1+z}=\frac{1}{4(1-z)}-\frac{1}{4(1+z)}$$
$$+\frac{1}{4}\ln\left|\frac{1+z}{1-z}\right|+C=\frac{z}{2(1-z^2)}+\frac{1}{4}\ln\left|\frac{1+z}{1-z}\right|+C$$
$$=\frac{\tan t}{2\cos t}+\frac{1}{4}\ln\left|\frac{1+\sin t}{1-\sin t}\right|+C.$$

We then return to the variable x: $\frac{1}{\cos t}=\sqrt{1+\tan^2 t}=\sqrt{x^2+1}$, $\sin t=\sqrt{1-\cos^2 t}=\frac{x}{\sqrt{x^2+1}}$, $\ln\left|\frac{1+\sin t}{1-\sin t}\right|=\ln\left|\frac{(1+\sin t)^2}{1-\sin^2 t}\right|=2\ln\left|\frac{1+\sin t}{\cos t}\right|=2\ln|x+\sqrt{x^2+1}|$. Finally,

$$\int \sqrt{1+x^2}\,dx=\frac{x}{2}\sqrt{x^2+1}+\frac{1}{2}\ln|x+\sqrt{x^2+1}|+C. \blacktriangle$$

In the general case, when it is required to find the integral of the form

$$\int R(x, \sqrt{ax^2+bx+c})\,dx,$$

where R is a rational function of its arguments x and $\sqrt{ax^2+bx+c}$, we use Euler's substitutions:

(1) if $a>0$, then $\sqrt{ax^2+bx+c}=t-\sqrt{a}x$;

(2) if $c>0$, then $\sqrt{ax^2+bx+c}=xt+\sqrt{c}$;

(3) if $b^2-4ac>0$, then $\sqrt{ax^2+bx+c}=(x-\alpha)t$, where α is one of the roots of the equation $ax^2+bx+c=0$. Consider an example in which Euler's substitutions are used for finding integrals.

Example 5. Compute the integral $\int \frac{dx}{\sqrt{-x^2-3x+4}}$.

△ Since $c=4$, we set $\sqrt{-x^2-3x+4}=xt+2$. Then $-x^2-3x+4=x^2t^2+4xt+4$, $x=-\frac{4t+3}{1+t^2}$, $dx=2\frac{2t^2+3t-2}{(1+t^2)^2}\,dt$, $\sqrt{-x^2-3x+4}=-\frac{2t^2+3t-2}{1+t^2}$. Thus,

$$\int \frac{dx}{\sqrt{-x^2-3x+4}}=-\int \frac{2dt}{1+t^2}=-2\arctan t+C.$$

Finally, we have:

$$\int \frac{dx}{\sqrt{-x^2-3x+4}} = -2\arctan\left(\frac{\sqrt{-x^2-3x+4}-2}{x}\right)+C. \blacktriangle$$

3. Integrals of Some Transcendental Functions. In applications, we often come across integrals of trigonometric functions. Consider the following cases of finding the integrals of such functions.

Let $R(u, v)$ be a rational function of the variables $u = \sin x$ and $v = \cos x$. Then the integral $\int R(\sin x, \cos x)\,dx$ is reduced to the integral of a rational function by the substitution $t = \tan\frac{x}{2}$. Indeed, since

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$x = 2\arctan t, \quad dx = \frac{2\,dt}{1+t^2},$$

we have

$$\int R(\sin x, \cos x)\,dx = 2\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{dt}{1+t^2}. \qquad (1)$$

Thus, with the help of the substitution $t = \tan\frac{x}{2}$, which is called the universal substitution, the given integral is reduced to the integral of a rational function with respect to the variable t.

Example 1. Compute the integral $\int \frac{dx}{3+5\cos x}$.

△ Setting $t = \tan\frac{x}{2}$, according to relation (1), we have:

$$\int \frac{dx}{3+5\cos x} = 2\int \frac{1}{\left(3+5\frac{1-t^2}{1+t^2}\right)}\frac{dt}{1+t^2} = \int \frac{dt}{4-t^2}$$

$$= \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| + C = \frac{1}{4}\ln\left|\frac{2+\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right| + C. \blacktriangle$$

Note that the process of finding integrals of functions $R(\sin x, \cos x)$ with the help of the universal substitution is not always con-

cise and simple (just because of the universality of the substitution). Therefore, if, for instance, $\sin x$ and $\cos x$ enter the expression of the function R only in even powers, then it is more expedient to make use of the substitution $t = \tan x$ since

$$\int R(\sin^2 x,\ \cos^2 x)\,dx = \int R\left(\frac{t^2}{1+t^2},\ \frac{1}{1+t^2}\right)\frac{dt}{1+t^2}. \qquad (2)$$

Example 2. Compute the integral $\int \tan^2 x\,dx$.

△ We set $t = \tan x$. Then $x = \arctan t$, $dx = \frac{dt}{1+t^2}$, and consequently,

$$\int \tan^2 x\,dx = \int t^2 \frac{dt}{1+t^2}$$
$$= \int \left(1 - \frac{1}{1+t^2}\right) dt = t - \arctan t + C = \tan x - x + C. \ ▲$$

Example 3. Compute the integral $\int \frac{dx}{1+8\cos^2 x}$.

△ Setting $t = \tan x$ and using formula (2), we get:

$$\int \frac{dx}{1+8\cos^2 x} = \int \frac{dt}{\left(1+8\frac{1}{1+t^2}\right)(1+t^2)} = \int \frac{dt}{9+t^2}$$
$$= \frac{1}{3}\arctan\frac{t}{3} + C = \frac{1}{3}\arctan\left(\frac{\tan x}{3}\right) + C. \ ▲$$

Sometimes, to find the integrals of functions $R(\sin x, \cos x)$, it is useful to apply the substitution $t = \cos x$ or $t = \sin x$. For instance, to compute the integrals $\int \sin^m x \cos^n x\,dx$ we may use the indicated substitutions if at least one of the numbers m and n is odd.

Example 4. Compute the integral $\int \sin^7 x \cos x\,dx$.

△ Setting $t = \sin x$, we get $dt = \cos x\,dx$. Consequently,

$$\int \sin^7 x \cos x\,dx = \int t^7\,dt = \frac{t^8}{8} + C = \frac{1}{8}\sin^8 x + C. \ ▲$$

Sometimes, when computing integrals of trigonometric functions, we have to transform the integrand in an appropriate way and then to use the basic table of simple integrals.

Example 5. Compute the integral $\int \sin^2 x\,dx$.

△ Since $\sin^2 x = \frac{1-\cos 2x}{2}$, we have

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{1}{4}\int \cos 2x\,d(2x)$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} + C. \ ▲$$

When computing integrals of the form $\int \sin mx \cos nx\,dx$, $\int \sin mx \sin nx\,dx$, and $\int \cos mx \cos nx\,dx$ it is convenient to use the following trigonometric formulas:

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

Example 6. Compute the integral $\int \sin 6x \cos 4x\,dx$.

△ Since $\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$, we have

$$\int \sin 6x \cos 4x\,dx = \frac{1}{2}\int \sin 2x\,dx + \frac{1}{2}\int \sin 10x\,dx$$
$$= -\frac{1}{4}\cos 2x - \frac{1}{20}\cos 10x + C. \blacktriangle$$

If the integrand is a product of $\sin mx$ (or $\cos mx$) by a polynomial or by the function e^{ax}, then for computing integrals of such functions we use the method of integration by parts (sometimes it should be used for several times).

Example 7. Compute the integral $\int x \sin 2x dx$.

△ We set $u = x$, $dv = \sin 2x\,dx = -\frac{1}{2}d\cos 2x$. Then

$$\int x \sin 2x\,dx = -\frac{1}{2}\int x\,d\cos 2x = -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx$$
$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + C. \blacktriangle$$

Example 8. Compute the integral $\int x^2 \cos x\,dx$.

△ We apply twice the formula for integration by parts:

$$\int x^2 \cos x\,dx = \int x^2 d\sin x = x^2 \sin x - \int \sin x\,dx^2$$
$$= x^2 \sin x - 2\int x \sin x\,dx = x^2 \sin x + 2\int x\,d\cos x$$
$$= x^2 \sin x + 2x\cos x - 2\int \cos x\,dx$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + C. \blacktriangle$$

Example 9. Compute the integral $\int e^{\alpha x} \sin 2x\, dx$.

△ We set $u = e^{\alpha x}$, $dv = \sin 2x\, dx = -\frac{1}{2} d \cos 2x$. Then

$$\int e^{\alpha x} \sin 2x\, dx = -\frac{1}{2} \int e^{\alpha x} d \cos 2x$$

$$= -\frac{1}{2} e^{\alpha x} \cos 2x + \frac{\alpha}{2} \int e^{\alpha x} \cos 2x\, dx. \qquad (3)$$

The last integral is also computed by parts:

$$\int e^{\alpha x} \cos 2x\, dx = \frac{1}{2} \int e^{\alpha x} d \sin 2x = \frac{1}{2} e^{\alpha x} \sin 2x - \frac{\alpha}{2} \int e^{\alpha x} \sin 2x\, dx.$$

Substituting this expression into equality (3), we get the equation with respect to the initial integral:

$$\int e^{\alpha x} \sin 2x\, dx = \frac{\alpha}{4} e^{\alpha x} \sin 2x - \frac{1}{2} e^{\alpha x} \cos 2x - \frac{\alpha^2}{4} \int e^{\alpha x} \sin 2x\, dx.$$

Hence we find that

$$\int e^{\alpha x} \sin 2x\, dx = \frac{e^{\alpha x} (\alpha \sin 2x - 2 \cos 2x)}{\alpha^2 + 4} + C. \ ▲$$

4. "Inexpressible" Integrals. On having considered some methods of computing indefinite integrals, we should like to note that not for every elementary function its antiderivative is an elementary function. If a certain antiderivative of an elementary function $f(x)$ is also an elementary function, then we say that the integral $\int f(x)\, dx$ is expressed in terms of elementary functions, or that this integral is computable (or "expressible"). But if an integral is not expressed in terms of elementary functions, we then say that it "cannot be found", or that it is "inexpressible". For instance, the functions

$$e^{-x^2}, \ \frac{\sin x}{x}, \ \frac{\cos x}{x}, \ \frac{1}{\ln x}, \ \sin (x^2), \ \sqrt{1 + x^3}$$

have antiderivatives but are not elementary functions. Therefore, the integrals

$$\int e^{-x^2} dx, \ \int \frac{\sin x}{x} dx, \ \int \frac{\cos x}{x} dx, \ \int \frac{dx}{\ln x},$$

$$\int \sin (x^2)\, dx, \ \int \sqrt{1 + x^3}\, dx$$

are said to be "inexpressible".

Exercises

1. Compute the following integrals:

(a) $\int \frac{dx}{x(x+a)}$; (b) $\int \frac{dx}{(x+a)(x+b)}$; (c) $\int \frac{x^4\,dx}{x^2+a}$;

(d) $\int \frac{x^5\,dx}{x^3-a^3}$; (e) $\int \frac{dx}{x^3+8}$; (f) $\int \frac{2x+7}{x^2+x-2}\,dx$;

(g) $\int \frac{(x+1)^3}{x^2-x}\,dx$, (h) $\int \frac{2x^2-5x+1}{x^3-2x^2+x}\,dx$, (i) $\int \frac{5x+2}{x^2+2x+10}\,dx$;

(j) $\int \frac{dx}{x^3-8}$; (k) $\int \frac{dx}{x^2+2x+5}$; (l) $\int \frac{dx}{3x^2-2x+4}$;

(m) $\int \frac{dx}{x^2+3x+1}$; (n) $\int \frac{du}{u^2-6u+5}$; (o) $\int \frac{x^5+x^4-8}{x^3-4x}\,dx$.

2. Compute the following integrals:

(a) $\int \frac{x+1}{\sqrt[3]{3x+1}}\,dx$; (b) $\int \frac{x\,dx}{\sqrt{2x+1}+1}$; (c) $\int x\sqrt{a-x}\,dx$;

(d) $\int \frac{x^3\,dx}{\sqrt{x^2+2}}$; (e) $\int \sqrt{a^2-x^2}\,dx$; (f) $\int x^2\sqrt{4-x^2}\,dx$;

(g) $\int \frac{x-1}{2x-1}\,dx$; (h) $\int \frac{\sqrt{x}\,dx}{x+1}$; (i) $\int \frac{x^2\,dx}{\sqrt{4-x^2}}$;

(j) $\int \frac{dx}{\sqrt{x^2+9}}$; (k) $\int \frac{dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$; (l) $\int \sqrt{\frac{1-x}{1+x}}\,dx$.

(3) Compute the given integrals:

(a) $\int \sin^3 x\,dx$; (b) $\int \tan^2 x\,dx$; (c) $\int \cot^3 x\,dx$;

(d) $\int \frac{dx}{\sin x}$; (e) $\int \frac{dx}{\cos x}$; (f) $\int \frac{dx}{\sin^3 x}$;

(g) $\int \frac{dx}{\cos^3 x}$; (h) $\int \cos^4 x\,dx$; (i) $\int \frac{dx}{\sin^4 x}$;

(j) $\int \tan^4 x\,dx$; (k) $\int \sin kx \sin mx\,dx$; (l) $\int x \sin mx\,dx$;

(m) $\int x \cos mx\,dx$; (n) $\int x^2 \sin mx\,dx$; (o) $\int e^x \cos mx\,dx$;

(p) $\int e^{-x} \sin 2x\,dx$; (q) $\int x \cos(2x+3)\,dx$.

Chapter 8

THE DEFINITE INTEGRALS

8.1. The Definite Integral and Its Properties

1. Problems Leading to the Notion of Definite Integral. Let $f(x)$, $x \in [a, b]$ be a nonnegative continuous function. The plane figure $AabB$ (Fig. 111) bounded by a segment of the axis of abscissas, segments of the vertical lines $x = a$ and $x = b$, and the graph of the given function $f(x)$ is called a curvilinear trapezoid. Let us find the area of this curvilinear trapezoid. To this end, we divide the closed interval $[a, b]$ into n subintervals of equal length $[a, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, b]$, using for this purpose the points

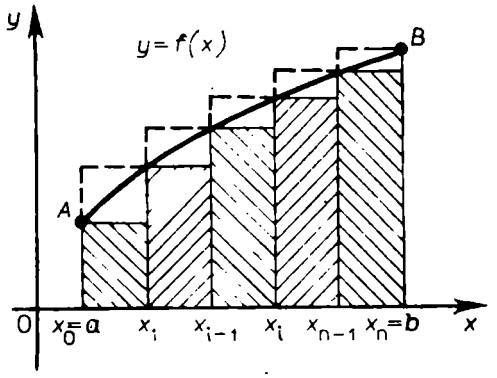

Fig. 111

$$x_i = a + \frac{b-a}{n} i, \quad i = 0, \quad 1, \quad \ldots, \quad n.$$

We then denote by m_i and M_i the least and the greatest value (respectively) of the function $f(x)$ on the interval $[x_{i-1}, x_i]$, where $i = 1, \ldots, n$. The curvilinear trapezoid $AabB$ is thus separated into n parts. Obviously, the area of the ith part is not less than $m_i(x_i - x_{i-1})$ and is not greater than $M_i(x_i - x_{i-1})$. Therefore the area of the entire curvilinear trapezoid $AabB$ is not less than the sum

$$m_1 \Delta x_1 + \ldots + m_n \Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $\Delta x_i = x_i - x_{i-1}$, and is not greater than the sum

$$M_1 \Delta x_1 + \ldots + M_n \Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i.$$

Denoting respectively these sums by s_n and S_n, we see that S_{AabB} satisfies the inequalities

$$s_n \leqslant S_{AabB} \leqslant S_n.$$

In these inequalities, s_n represents the area of the stepped figure contained in the given curvilinear trapezoid, and S_n the area of the stepped figure containing the given curvilinear trapezoid (shown in Fig. 111). It is intuitively clear that if the interval $[a, b]$ is divided into

sufficiently small subintervals, that is, if n is sufficiently large, then the areas of these figures differ but slightly from each other as also from the area of the curvilinear trapezoid for which we give the following definition:

Definition. Let $f(x)$, $x \in [a, b]$ be a continuous nonnegative function. Then, if the limits of the sequences $\{s_n\}$ and $\{S_n\}$ exist and are equal to each other, their common value is called the area of the curvilinear trapezoid.

Note. In each subinterval $[x_{i-1}, x_i]$ we choose an arbitrary point c_i, $x_{i-1} \leqslant c_i \leqslant x_i$. For this point the following relation will be valid:

$$m_i \leqslant f(c_i) \leqslant M_i, \tag{1}$$

where $i = 1, \ldots, n$. We multiply each of the inequalities in relation (1) by $\Delta x_i = x_i - x_{i-1}$ and add together the obtained inequalities. Then we get the following relation:

$$s_n \leqslant \sum_{i=1}^{n} f(c_i)\,\Delta x_i \leqslant S_n, \tag{2}$$

where the sum

$$f(c_1)\,\Delta x_1 + \ldots + f(c_n)\,\Delta x_n = \sum_{i=1}^{n} f(c_i)\,\Delta x_i \tag{3}$$

is called the integral sum of the function $f(x)$ on the interval $[a, b]$. Passing to the limit in inequality (2), as $n \to \infty$, we get that the limit of integral sum (3) exists, is independent of the choice of the points c_i, and is equal to the area of the figure $AabB$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = S_{AabB}. \tag{4}$$

We can prove that formula (4) also remains valid if the interval $[a, b]$ is subdivided into n subintervals of an arbitrary length, but such that the greatest of these lengths tends to zero as $n \to \infty$.

We encounter the necessity of computing the limit of integral sums in problems of various physical nature, for instance, when introducing the notion of work done by a force. Let a material point move in a straight line under the action of a constant force F. The position of the material point is characterized by the x-coordinate. The work A done by the force F is computed by the formula

$$A = F \cdot \Delta x, \tag{5}$$

where Δx is the displacement of the material point. Let now the material point move along the x-axis under the action of a variable force $f(x)$, at the beginning of motion it being at the point $x = a$, and at the end of motion at the point $x = b$. With the help of the points

$x_i = a + \frac{b-a}{n} i$ we divide the interval $[a, b]$ into n parts $[x_{i-1}, x_i]$ of equal length $\Delta x_i = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$ the work done by the force can be approximately computed by formula (5), that is, it can be assumed to be equal to $f(c_i)\,\Delta x_i$, where c_i is some point belonging to the subinterval $[x_{i-1}, x_i]$. Then the work of the force done on the interval $[a, b]$ will be computed approximately by the formula

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

The shorter the subintervals $[x_{i-1}, x_i]$, the higher the accuracy of approximation. Therefore, passing to the limit as $n \to \infty$, we get the equality

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i,$$

coinciding by form with equality (4).

2. The Notion of Definite Integral. Integrability Conditions. The examples considered in the preceding subsection show that problems of various nature are reduced to the problem on finding the limit of an integral sum.

Definition. Let there be given a function $f(x)$, $x \in [a, b]$. If the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists and is independent of the choice of points c_i, then the function $f(x)$, is said to be integrable on the interval $[a, b]$, and the limit is called the definite integral of the function $f(x)$ over the interval $[a, b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

This designation is read: "The integral from a to b of the function $f(x)$ with respect to dx" or, briefer, "the integral from a to b of $f(x)\,dx$". The symbol $\int$ is called the integral sign, the function $f(x)$ the integrand, x the variable of integration, and the expression $f(x)\,dx$ the element of integration. The numbers a and b are called (respectively) the lower and upper limits of integration. (It is unfortunate that the word "limit" is used in this connection. The limits of

integration have nothing to do with the limits considered in Chapter 5.)

Thus, according to the definition,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

The definite integral is independent of what letter denotes the variable of integration. For instance,

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du.$$

We can prove that the definite integral

$$\int_a^b f(x)\,dx$$

exists if the function $f(x)$ is bounded on the closed interval $[a, b]$ and has a finite number of points of discontinuity on this interval.

For a function to be integrable on an interval, it is necessary that the function be bounded on this interval. Bounded functions encountered in practice are, as a rule, integrable on any interval they are defined on. However, there are functions which are bounded on a closed interval, but are not integrable on this interval. Dirichlet's function is an example of such a bounded nonintegrable on [0, 1] function. It is equal to 1 at rational points and to zero at irrational points. Consequently, if in integral sums we choose irrational points as c_i, then

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} 0 \cdot x_i = 0,$$

and therefore the limit of these sums is equal to 0 as $n \to \infty$. And if rational points are chosen as c_i, then

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1,$$

and therefore the limit of these sums is equal to 1. Thus, in the case of Dirichlet's function the limit of integral sums on the interval [0, 1] depends on the choice of points c_i. This just means that the Dirichlet function is not integrable on the interval [0, 1].

3. Basic Properties of Definite Integrals.

1. For any real number α

$$\int_a^b \alpha\,dx = \alpha\,(b-a). \tag{1}$$

□ For any integral sum of the function $f(x) = \alpha$, $x \in [a, b]$, we have:

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} \alpha\,\Delta x_i = \alpha \cdot \sum_{i=1}^{n} \Delta x_i = \alpha(b-a),$$

and therefore the limit of the integral sums as $n \to \infty$ is equal to $\alpha(b-a)$. ■

2. Let a function $f(x)$ be integrable on the closed interval $[a, b]$, then for any real number α the function $\alpha f(x)$ is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx, \tag{2}$$

that is, a constant factor can be taken outside the integral sign.

□ Since any integral sum of the function $\alpha f(x)$ has the form:

$$\sum_{i=1}^{n} \alpha f(c_i)\,\Delta x_i = \alpha \sum_{i=1}^{n} f(c_i)\,\Delta x_i,$$

we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha f(c_i)\,\Delta x_i = \alpha \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \alpha \int_a^b f(x)\,dx,$$

which just means that the function $\alpha f(x)$ is integrable on $[a, b]$, and formula (2) holds true. ■

3. If functions $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$, then their sum and difference are also integrable on $[a, b]$ and

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx, \tag{3}$$

that is, the integral of the sum (difference) is equal to the sum (difference) of the integrals.

□ Indeed,

$$\int_a^b (f(x) \pm g(x))\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (f(c_i) \pm g(c_i))\,\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i \pm \lim_{n\to\infty} \sum_{i=1}^{n} g(c_i)\,\Delta x_i = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx. \blacksquare$$

4. If functions $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$ and

$$f(x) \leqslant g(x), \tag{4}$$

then the following inequality is valid:

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx. \tag{5}$$

□ Indeed, from inequality (4) it follows that for any integral sums of the functions $f(x)$ and $g(x)$ the following inequality holds:

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i \leqslant \sum_{i=1}^{n} g(c_i)\,\Delta x_i.$$

Passing to the limit in the last inequality as $n \to \infty$, we get inequality (5). ■

5. Let a function $f(x)$ be integrable on the closed inerval $[a, b]$, then it is integrable on any subinterval contained in $[a, b]$. In addition, if the function $f(x)$ is integrable both on $[a, c]$ and on $[c, b]$ then it is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{6}$$

We leave this property without proof.

6. Let on the interval $[a, b]$ the inequality

$$m \leqslant f(x) \leqslant M$$

be valid, where m and M are some numbers, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a). \tag{7}$$

□ The validity of this property follows from properties 1 and 4. ■

Note. In the preceding subsection we considered the definite integral for the case when the lower limit of integration is smaller than the upper one. In practice such a restriction causes some inconveniences. In order to get rid of this restriction, we introduce the following definition.

If $a = b$, then $\int_a^b f(x)\,dx = 0$,

and if $a > b$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

The last formula means that if the upper and lower limits of the definite integral are interchanged, the integral only changes its sign. We can prove that for such integrals the above properties are valid, only the inequaliy sign in formulas (5) and (7) will reverse.

Exercises

1. Compute the area of a curvilinear trapezoid under the graph of the function $y = e^x$ on the interval $[0, 1]$.

2. Prove that the area of a curvilinear trapezoid determined by the portion of the parabola $y = 1 - x^2$ $(0 \leqslant x \leqslant 1)$ is equal to $\frac{2}{3}$.

3. Prove the following corollary from property 4: if on the interval $[a, b]$ the functions $f(x)$ and $g(x)$ are integrable and $|f(x)| \leqslant g(x)$, then

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b g(x)\,dx.$$

8.2. Main Theorems on the Definite Integral

1. Mean-value Theorem. For a continuous function the following important theorem holds true. It is known as the mean-value theorem for the definite integral.

Theorem 8.2.1. *Let a function $f(x)$ be continuous on the interval $[a, b]$. Then there exists on $[a, b]$ a point c such that*

$$\int_a^b f(x)\,dx = f(c)(b-a). \tag{1}$$

□ If $a = b$, then formula (1) is obvious. Let $a < b$. We denote by m and M the least and greatest values of the function $f(x)$ on $[a, b]$. Then $m \leqslant f(x) \leqslant M$ for any $x \in [a, b]$, and therefore

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

Dividing termwise this inequality by $b - a$, we get:

$$m \leqslant \frac{1}{b-a} \cdot \int_a^b f(x)\,dx \leqslant M.$$

Hence there follows the validity of formula (1). Indeed, since $f(x)$ is continuous on $[a, b]$, it takes on any value from the interval $[m, M]$ and therefore there exists a point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

If now $a > b$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = -f(c)(a-b) = f(c)(b-a),$$

where $c \in [b, a]$, and the theorem has been proved completely. ■

2. Definite Integrals with a Variable Upper Limit. Let there be given a function $f(x)$ continuous on the interval $[a, b]$. Then, as was noted above, it is integrable on any subinterval $[a, x]$, where $x \in [a, b]$. Consider the function

$$\Phi(x) = \int_a^x f(z)\,dz, \quad x \in [a, b]. \tag{1}$$

This function is called an integral with a variable upper limit. In formula (1), the variable of integration is denoted by the letter z, since the letter x denotes the upper limit of integration (here it is the independent variable, that is, the argument of the function Φ).

Theorem 8.2.2. *If a function $f(x)$ is continuous on the interval $[a, b]$, then the function $\Phi(x)$ has a derivative on $[a, b]$, and $\Phi'(x) = f(x)$, that is,*

$$\frac{d}{dx}\int_a^x f(z)\,dz = f(x). \tag{2}$$

□ From the representation of the function $\Phi(x)$ and property 5 of the integral (see the preceding section) it follows that

$$\Phi(x) - \Phi(x_0) = \int_a^x f(z)\,dz - \int_a^{x_0} f(z)\,dz = \int_{x_0}^x f(z)\,dz$$

for any x and x_0 from $[a, b]$. Then, by mean value theorem, we find that

$$\Phi(x) - \Phi(x_0) = f(c)(x - x_0),$$

where $c \in [x_0, x]$ if $x_0 < x$, and $c \in [x, x_0]$ if $x < x_0$. Thus, for any $x \neq x_0$ there is a c, lying between x and x_0, such that

$$\frac{\Phi(x) - \Phi(x_0)}{x - x_0} = f(c).$$

Since $f(x)$ is a function continuous on the interval $[a, b]$ and, consequently, also at the point $x_0 \in [a, b]$, we have

$$\Phi'(x_0) = \lim_{x \to x_0} \frac{\Phi(x) - \Phi(x_0)}{x - x_0} = \lim_{x \to x_0} f(c) = f(x_0).$$

Obviously, at the point $x_0 = a$ the function $\Phi(x)$ will have a right-hand derivative and at the point $x_0 = b$ a left-hand derivative. ■

It follows from this theorem that any continuous function has an antiderivative which is a definite integral with a variable upper limit of the given function.

3. The Newton-Leibniz Formula.

Theorem 8.2.3. *If a function $f(x)$ is continuous on the interval $[a, b]$, and the function $F(x)$ is an antiderivative of $f(x)$ on $[a, b]$, then the following formula is valid:*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

This is the Newton-Leibniz formula.

□ According to the theorem on differentiation of the integral with respect to the upper limit, the function

$$\Phi(x) = \int_a^x f(u)\,du$$

is an antiderivative of the function $f(x)$ on $[a, b]$. Since the function $F(x)$ is also an antiderivative of $f(x)$ on $[a, b]$, the difference $\Phi(x) - F(x)$ is equal to a certain constant C over the entire interval $[a, b]$, that is

$$\Phi(x) = F(x) + C.$$

Setting here first $x = a$ then $x = b$, we get

$$\Phi(a) = F(a) + C,$$
$$\Phi(b) = F(b) + C.$$

Since $\Phi(a) = 0$ and $\Phi(b) = \int_a^b f(x)\,dx$, we have $C = -F(a)$, and, consequently,

$$\Phi(b) = F(b) - F(a). \ ■$$

For brevity, instead of the difference $F(b) - F(a)$ we often write $F(x)\Big|_a^b$, and then formula (1) takes the form:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b. \tag{2}$$

The Newton-Leibniz formula enables us to compute definite integrals whenever at least one antiderivative of the integrand is known. The methods of finding antiderivatives were considered in Chapter 7.

Let us solve a few examples of computing definite integrals with the help of the Newton-Leibniz formula.

Example 1. Evaluate the integral $\int_0^1 x^3\,dx$.

△ By the Newton-Leibniz formula, we have

$$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}\,. \blacktriangle$$

Example 2. Evaluate the integral $\int_{-1}^1 (x^2 + \sin x - 7x^{10})\,dx$.

△ As is known, the integral of a sum (difference) is equal to the sum (difference) of the integrals, therefore we have:

$$\int_{-1}^1 (x^2 + \sin x - 7x^{10})\,dx = \int_{-1}^1 x^2\,dx + \int_{-1}^1 \sin x\,dx - 7\int_{-1}^1 x^{10}\,dx$$

$$= \frac{x^3}{3}\Big|_{-1}^1 - \cos x\Big|_{-1}^1 - \frac{7}{11}x^{11}\Big|_{-1}^1$$

$$= \frac{1}{3}(1+1) - (\cos 1 - \cos(-1))$$

$$- \frac{7}{11}(1-(-1)) = -\frac{20}{33}\,. \blacktriangle$$

Example 3. Evaluate the integral $\int_0^2 f(x)\,dx$, where $f(x) =$

$$\begin{cases} e^x, & x \in [0,\ 1), \\ 3x^2, & x \in [1,\ 2]. \end{cases}$$

△ Since the integral from 0 to 2 can be represented as the sum of the integrals from 0 to 1 and from 1 to 2, we have

$$\int_0^2 f(x)\,dx = \int_0^1 e^x\,dx + \int_1^2 3x^2\,dx = e^x\Big|_0^1 + x^3\Big|_1^2$$

$$= e - 1 + 8 - 1 = e + 6. \blacktriangle$$

Exercises

Using the Newton-Leibniz formula, compute the following integrals:

(a) $\int_1^2 (x+1)^2\,dx$; (b) $\int_2^5 \left(x^2 + \frac{1}{x^3}\right)dx$; (c) $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$;

(d) $\int_0^a (x^2-ax)\,dx$; (e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\cos^2 x}$; (f) $\int_1^2 e^x\left(1-\frac{e^{-x}}{x^2}\right)dx$;

(g) $\int_0^2 g(x)\,dx$, where $g(x)=x$ for $0\leqslant x\leqslant 1$ and $g(x)=(2-x)$ for $1\leqslant x\leqslant 2$; (h) $\int_1^3 f(x)\,dx$, where $f(x)=(3-x)$ for $1\leqslant x\leqslant 2$ and $f(x)=-(x-4)^2$ for $2<x\leqslant 3$.

8.3. Methods of Computing Definite Integrals

1. The Method of Change of the Variable of Integration (the Method of Substitution). When computing definite integrals, the same as indefinite ones, the method of substitution or the method of change of the variable of integration is widely used.

Theorem 8.3.1. *Let a function $f(x)$ be continuous at any point $x=\varphi(t)$, where $t\in[\alpha,\beta]$, and let $a=\varphi(\alpha)$ and $b=\varphi(\beta)$. Then, if the function $\varphi(t)$ has a continuous derivative, the following formula holds:*

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt. \tag{1}$$

Formula (1) is called the formula for changing the variable of integration in definite integrals.

□ By hypothesis, $f(x)$ is a continuous function, therefore it has an antiderivative. Let us denote it by $F(x)$. Then the composite function $F(\varphi(t))$ will be an antiderivative of the function $f(\varphi(t))\varphi'(t)$. Further, we compute the integrals of the functions $f(x)$ and $f(\varphi(t))\,\varphi'(t)$ using the Newton-Leibniz formula:

$$\int_a^b f(x)\,dx=F(b)-F(a), \tag{2}$$

$$\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt=F(\varphi(\beta))-F(\varphi(\alpha)). \tag{3}$$

By hypothesis, $b=\varphi(\beta)$ and $a=\varphi(\alpha)$, and therefore the right-hand sides in (2) and (3) coincide. Consequently, their left-hand sides are also equal, that is, formula (1) holds true. ■

Formula (1) can be written in the following form:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,d\varphi(t). \tag{4}$$

Formula (4) is applied both from left to right and from right to left. In the first case, instead of x, we write $\varphi(t)$, while in the second case, instead of $\varphi(t)$, we write x (or y or z, etc.). In both cases, we have to change the limits of integration in a proper way.

Example 1. Evaluate the integral $\int_0^{\frac{\pi}{4}} \sin 2x\,dx$.

△ We introduce a new variable $t=2x$. Then $dx=\frac{1}{2}dt$, the lower limit of integration will become $t(0)=2\cdot 0=0$, and the upper $t\left(\frac{\pi}{4}\right)=2\cdot\frac{\pi}{4}=\frac{\pi}{2}$. Finally,

$$\int_0^{\frac{\pi}{4}} \sin 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin t\,dt = -\frac{1}{2}\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}(0-1)=\frac{1}{2}. \ ▲$$

Example 2. Evaluate the integral $\int_1^2 x\cos x^2\,dx$.

△ We introduce a new variable $t=x^2$. Then $dt=2x\,dx$, the lower limit $t(1)=1$, upper limit $t(2)=4$. Finally, we find:

$$\int_1^2 x\cos x^2\,dx = \frac{1}{2}\int_1^4 \cos t\,dt = \frac{1}{2}\sin t\Big|_1^4 = \frac{\sin 4-\sin 1}{2}. \ ▲$$

Example 3. Evaluate the integral $\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}}$.

△ We introduce a new variable $t=1+x^2$. Then $dt=2x\,dx$, the lower limit $t(0)=1$, the upper limit $t(\sqrt{3})=4$. Consequently,

$$\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\int_1^4 \frac{dt}{\sqrt{t}} = \sqrt{t}\Big|_1^4 = \sqrt{4}-\sqrt{1}=1. \ ▲$$

Example 4. Evaluate the integral $\int_0^{\pi} e^{\cos x}\sin x\,dx$.

△ We introduce a new variable $t=\cos x$. Then $dt=-\sin x\,dx$, the lower limit $t(0)=1$, the upper limit $t(\pi)=-1$.

Consequently,

$$\int_0^{\pi} e^{\cos x} \sin x \, dx = -\int_1^{-1} e^t \, dt = \int_{-1}^{1} e^t \, dt = e^t \Big|_{-1}^{1} = e - e^{-1}. \blacktriangle$$

Example 5. Evaluate the integral $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$.

△ We made the substitution $\tan \frac{x}{2} = t$, that is, $x = 2 \arctan t$. From trigonometric formulas it follows that

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2},$$

and therefore

$$\frac{1}{1+\cos x} = \frac{1+t^2}{1+t^2+1-t^2} = \frac{1+t^2}{2}.$$

We now compute dx:

$$dx = 2 (\arctan t)' \, dt = \frac{2dt}{1+t^2}.$$

Further, we find the new limits of integration. Since $\tan \frac{0}{2} = 0$, and $\tan \left(\frac{1}{2} \cdot \frac{\pi}{2} \right) = 1$, the new limits of integration will be 0 and 1. Thus

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x} = \int_0^1 \frac{1+t^2}{2} \cdot \frac{2dt}{1+t^2} = \int_0^1 dt = t \Big|_0^1 = 1. \blacktriangle$$

2. The Method of Integration by Parts. As is known, the formula of integration by parts for indefinite integrals is proved by integrating the equality

$$(uv)' = uv' + vu'. \tag{1}$$

The formula of integration by parts for definite integrals is proved in a similar way.

Theorem 8.3.2. *Let functions $u(x)$ and $v(x)$ have continuous derivatives on the interval $[a, b]$. Then the following formula is valid:*

$$\int_a^b u(x) v'(x) \, dx = u(x) v(x) \Big|_a^b - \int_a^b v(x) u'(x) \, dx. \tag{2}$$

□ From equality (1) it follows that

$$\int_a^b (uv)'\,dx = \int_a^b uv'\,dx + \int_a^b vu'\,dx.$$

As is known, the integral of $(uv)'$ is equal to $uv\Big|_a^b$, therefore

$$uv\Big|_a^b = \int_a^b uv'\,dx + \int_a^b vu'\,dx.$$

Hence follows the validity of formula (2). ■

The formula of integration by parts enables us to reduce the computation of one integral to that of another integral. It goes without saying, that, by applying this formula, we try to succeed in replacing the initial integral by a simpler one or by one which is more convenient for study.

Formula (2) can be written in a different way:

$$\int_a^b u(x)\,dv(x) = u(x)\,v(x)\Big|_a^b - \int_a^b v(x)\,du(x). \tag{3}$$

Example 1. Evaluate the integral $\int_1^e \ln x\,dx$.

△ We make use of the formula for integration by parts. For this purpose, setting $u = \ln x$, $dv = dx$, we get:

$$\int_1^e \ln x\,dx = x\ln x\Big|_1^e - \int_1^e x\,d\ln x = e - \int_1^e x\frac{1}{x}\,dx = e - \int_1^e dx = 1. \ ▲$$

Example 2. Evaluate the integral $\int_0^\pi x^2\cos x\,dx$.

△ Applying formula (3) and setting $u = x^2$, $dv = \cos x\,dx = d\sin x$, we find $\int_0^\pi x^2\cos x\,dx = \int_0^\pi x^2 d\sin x = x^2\sin x\Big|_0^\pi - 2\int_0^\pi x\sin x\,dx =$ $2\int_0^\pi x\,d\cos x$. To the obtained integral we apply the formula for integration by parts once again:

$$2\int_0^\pi x\,d\cos x = 2x\cos x\Big|_0^\pi - 2\int_0^\pi \cos x\,dx = -2\pi - 2\sin x\Big|_0^\pi = -2\pi. \ ▲$$

Exercises

1. Evaluate the following integrals:

(a) $\int_0^{\frac{\pi}{4}} \sin 4x\,dx$; (b) $\int_0^{\pi} \cos 2x\,dx$; (c) $\int_0^{1} e^{3x}\,dx$;

(d) $\int_{-1}^{1} x(2-x)^2\,dx$; (e) $\int_1^{2} \frac{e^x}{e^x-1}\,dx$; (f) $\int_1^{4} \frac{dx}{\sqrt{5-x}}$;

(g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x\,dx}{\sin^3 x}$; (h) $\int_0^{1} \tan 3x\,dx$; (i) $\int_a^{a\sqrt{3}} \frac{dx}{a^2+x^2}$;

(j) $\int_4^{9} \frac{dx}{\sqrt{x}-1}$; (k) $\int_0^{4} \frac{dx}{1+\sqrt{2x+1}}$; (l) $\int_{-1}^{2} x \sin x^2\,dx$;

(m) $\int_0^{1} \frac{dx}{e^x+1}$; (n) $\int_0^{1} \ln(1+x)\,dx$.

2. Evaluate the following integrals:

(a) $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$; (b) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$; (c) $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$;

(d) $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$; (e) $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$; (f) $\int_0^{\frac{\pi}{4}} \tan^3 x\,dx$;

(g) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x\,dx$; (h) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x\,dx$; (i) $\int_0^{\frac{\pi}{2}} x \cos x\,dx$;

(j) $\int_0^{\pi} x^3 \sin x\,dx$; (k) $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$; (l) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x\,dx}{\sin^2 x}$;

(m) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

3. Evaluate the following integrals:

(a) $\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{4-x^2}}$; (b) $\int_0^1 \frac{e^x\,dx}{1+e^{2x}}$; (c) $\int_1^5 \frac{x\,dx}{\sqrt{4x+5}}$;

(d) $\int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$; (e) $\int_1^6 \sqrt{3+x}\,dx$; (f) $\int_4^9 \frac{\sqrt{u}-1}{\sqrt{u}+1}\,du$;

(g) $\int_0^1 \frac{u\,du}{(u^2+1)^2}$; (h) $\int_1^2 \frac{e^{\frac{1}{x}}\,dx}{x^2}$; (i) $\int_2^3 \frac{dx}{2x^2+3x-2}$;

(j) $\int_0^1 \frac{dx}{x^2+4x+5}$; (k) $\int_{-\frac{1}{2}}^1 \frac{dx}{\sqrt{8+2x-x^2}}$; (l) $\int_0^1 xe^{-x}\,dx$;

(m) $\int_1^e \ln^3 x\,dx$; (n) $\int_0^1 x \ln(x^2+1)\,dx$; (o) $\int_0^a \sqrt{a^2-x^2}\,dx$.

8.4. Approximate Computation of the Definite Integral

In practice, it is often required to compute the definite integrals of functions for which one fails to find antiderivatives. In such cases, we confine ourselves to finding only an approximate value of the integral under consideration. Even if an antiderivative is known, it is sometimes more convenient to find an approximate value of an integral by the formulas for approximate integration. In the present section, we shall consider three most widely used formulas for approximate evaluation of definite integrals: the rectangle formula, trapezoid formula, and parabola formula (or Simpson's formula).

1. The Rectangle Formula (The Rectangle Rule). Suppose it is required to compute the integral of the function $f(x)$ from a to b. We divide the interval of integration $[a, b]$, by the points $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, \ldots, n$, into n equal parts (subintervals). Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(c_i), \qquad (1)$$

where $c_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$, is some point of the ith subinterval. From the definition of the limit it follows that with a sufficiently large n, we may take the integral sum for an approximation to the

desired integral:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \cdot \sum_{i=1}^{n} f(c_i). \qquad (2)$$

In formula (2), points c_i are still chosen arbitrarily. For simplicity, we often take the left-hand or right-hand endpoints of subintervals $[x_{i-1}, x_i]$ as points c_i, that is, $c_i = x_{i-1}$ or $c_i = x_i$. As a result, we get simple formulas for approximate computation of definite integrals

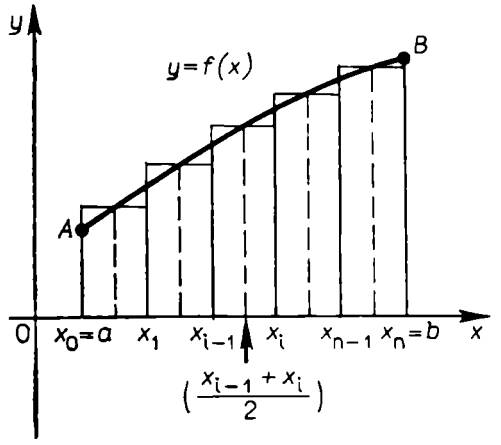

Fig. 112

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1}) \qquad (3)$$

or

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f(x_i), \qquad (4)$$

where $x_i = a + \frac{b-a}{n}\, i \left(x_{i-1} = a + \frac{b-a}{n}\,(i-1)\right)$, $i = 1, \ldots, n$. Formulas (3) and (4) are called the formulas of left- and right-hand rectangles, respectively.

We can prove that the error of formula (2) will abruptly decrease if the midpoints of subintervals $[x_{i-1}, x_i]$ are taken as points c_i, that is, if we set

$$c_i = \frac{x_{i-1}+x_i}{2} = a + \frac{b-a}{n}\left(i - \frac{1}{2}\right).$$

As a result, formula (2) takes the form:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right). \qquad (5)$$

Formula (5) is called the rectangle formula. For a given value of n formula (5) ensures a higher accuracy of computation of the integral $\int_b^a f(x)\,dx$, than formulas (3) and (4), therefore in what follows we shall investigate only this formula. For nonnegative functions $f(x)$ formula (5) means (see Fig. 112) that the integral over the subinterval $[x_{i-1}, x_i]$, equal to the area of the corresponding curvilinear

trapezoid, is replaced by the area of the rectangle (hence the name of the method) with the same base and altitude equal to $\left(\frac{x_{i-1}+x_i}{2}\right)$.

Let us work out a program in the BASIC language for approximate computation of integrals by the rectangle formula. As an example, we consider the integral

$$\int_0^1 \sin(x^2)\,dx. \tag{6}$$

This is an "inexpressible" integral (the antiderivative of the integrand $\sin(x^2)$ is not an elementary function), and therefore it cannot be computed by the Newton-Leibniz formula. We apply rectangle formula (5) to compute the given integral:

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{n}\sum_{i=1}^{n} \sin\left(\frac{i-\frac{1}{2}}{n}\right)^2$$

$$= \frac{1}{n}\left[\sin\left(\frac{1}{2n}\right)^2 + \sin\left(\frac{3}{2n}\right)^2 + \ldots + \sin\left(\frac{n-\frac{1}{2}}{n}\right)^2\right]. \tag{7}$$

For sufficiently large n ($n = 100$, 1000, etc.) computations by formula (7) can be practically performed only with the help of a computer. The program of computations by formula (7) in BASIC has the form:

```
1 INPUT N
2 LET H = 1/N
3 LET X = H/2
4 LET S = 0
5 FOR I = 1 TO N
6 LET S = S + SIN (X * X)
7 LET X = X + H
8 NEXT I
9 LET F = H * S
10 PRINT F
11 END
```

where n is denoted by the capital letter N, the step of increase in the argument $\frac{1}{n}$ by H, running value of the sum (its initial value is equal to zero) by S, running value of the argument (the initial value is equal to $\left(\frac{1}{2n} = \frac{H}{2}\right)$ by X, the value of the integral by F. Let us con-

sider the program in more detail. The computer performs computations according to the program passing from one line to another in the increasing order of their numbers. In accordance with the first line of the program, the sign? appears on the television screen of the personal computer, after which we introduce a concrete numerical value N with the help of the keyboard. Then the computer executes the program of computations independently. The initial values of the argument X and sum S are set in the third and fourth lines. Lines 5-8 are repeated cyclically N times, so that a new term SIN (X * X) is added each time to the value of the sum S (that is, $\sin(x^2)$), after which the argument X increases by one and the same magnitude $H = \frac{1}{N}$ (that is, by $\frac{1}{n}$). Thereby the sum in the brackets in formula (7) will be computed. In line 9 we multiply it by H and get the approximate value of the integral F, which is brought out to the television screen for observation with the help of the operation written in line 10. The eleventh line indicates that the program is completed. The results of computations according to the given program will be considered below.

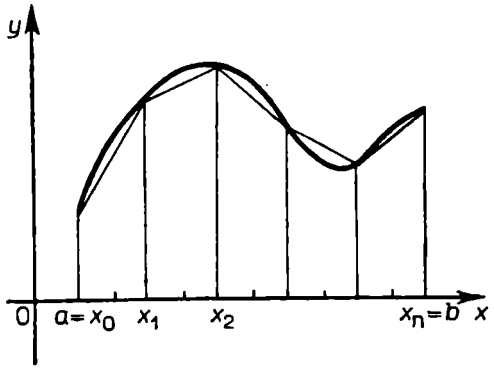

Fig. 113

2. The Trapezoid Formula (The Trapezoid Rule). If the area of a curvilinear trapezoid on the subinterval $[x_{i-1}, x_i]$ is replaced by the area of the corresponding trapezoid which is equal to (Fig. 113)

$$\frac{b-a}{n} \cdot \frac{f(x_{i-1})+f(x_i)}{2},$$

then we get the formula

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}. \qquad (1)$$

Formula (1) is called the trapezoid formula for approximate computation of definite integrals. Since

$$\sum_{i=1}^{n} f(x_{i-1}) = f(a)+f(x_1)+\ldots+f(x_{n-1}),$$

$$\sum_{i=1}^{n} f(x_i) = f(x_1)+\ldots+f(x_{n-1})+f(b),$$

we have

$$\sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2} = \frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i).$$

Thus, the trapezoid formula can be written in the following form, which is convenient for computations:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a+\frac{b-a}{n}\,i\right)\right]. \quad (2)$$

Let us now apply trapezoid formula (2) to computation of integral (6) from the preceding subsection:

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{n}\left[\frac{\sin(0^2)+\sin(1^2)}{2} + \sin\left(\frac{1}{n}\right)^2 + \ldots + \sin\left(\frac{n-1}{n}\right)^2\right]. \quad (3)$$

The program for computing the integral by the trapezoid formula in BASIC undergoes just insignificant and obvious changes as compared with the program of computations by the rectangle formula:

```
1 INPUT N
2 LET H = 1/H
3 LET X = H
4 LET S = 0
5 FOR I = 1 TO N - 1
6 LET S = S + SIN (X * X)
7 LET X = X + H
8 NEXT I
9 LET F = H * (S + (SIN (0) + SIN (1))/2)
10 PRINT F
11 END
```

The results of these computations will be considered below.

3. The Parabola Formula (Simpson's Formula). Let us replace the area of original curvilinear trapezoid on the subinterval $[x_{i-1}, x_i]$ by the area of the curvilinear parabolic trapezoid formed by the graph of the parabola $y = a_i x^2 + b_i x + c_i$, $x \in [x_{i-1}, x_i]$ passing through three points of the graph of the given function $f(x)$ with ab-

scissas x_{i-1}, $\frac{x_{i-1}+x_i}{2}$ and x_i, $i = 1, \ldots, n$:

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1}),$$
$$a_i \left(\frac{x_{i-1}+x_i}{2}\right)^2 + b_i \left(\frac{x_{i-1}+x_i}{2}\right) + c_i = f\left(\frac{x_{i-1}+x_i}{2}\right),$$
$$a_i x_i^2 + b_i x_i + c_i = f(x_i). \qquad (1)$$

The area of the ith curvilinear parabolic trapezoid is equal to

$$S_i = \int_{x_{i-1}}^{x_i} (a_i x^2 + b_i x + c_i)\, dx = \left(\frac{a_i x^3}{3} + \frac{b_i x^2}{2} + c_i x\right)\Big|_{x_{i-1}}^{x_i}$$
$$= \Delta x_i \left[\frac{a_i}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2) + \frac{b_i}{2}(x_{i-1} + x_i) + c_i\right],$$

where $\Delta x_i = x_i - x_{i-1}$. From formulas (1) we find that

$$\frac{f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)}{6} = \frac{a_i}{3}(x_{i-1}^2 + x_{i-1}x_i + x_i^2)$$
$$+ \frac{b_i}{2}(x_{i-1} + x_i) + c_i.$$

Then

$$S_i = \Delta x_i \cdot \frac{f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)}{6},$$

and, consequently,

$$\int_a^b f(x)\, dx \approx \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_{i-1}) + 4f\left(\frac{x_{i-1}+x_i}{2}\right) + f(x_i)}{6}. \qquad (2)$$

Formula (2) is called the parabola formula (or Simpson's formula) for approximate computation of definite integrals. In the preceding subsection it was shown that

$$\sum_{i=1}^{n} (f(x_{i-1}) + f(x_i)) = f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i),$$

and the parabola formula can be written in the form convenient for practical computations:

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6n}\left[f(a) + f(b) + 4\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) + 2\sum_{i=1}^{n-1} f(x_i)\right]. \qquad (3)$$

Applying this formula to integral (6) in subsection 1, we get:

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{6n}\left[\sin(0^2)+\sin(1^2)+4\sin\left(\frac{1}{2n}\right)^2+2\sin\left(\frac{1}{n}\right)^2\right.$$

$$\left.+\cdots+4\sin\left(\frac{2n-1}{2n}\right)^2\right]. \qquad (4)$$

The program of computations by formula (4) is analogous to the previous programs and is given without detailed explanations:

```
1 INPUT N
2 LET H = 1/(2 * N)
3 LET X = H
4 LET S = 0
5 FOR I = 1 TO N
6 LET S = S + 4 * SIN (X * X)
7 LET X = X + H
8 LET S = S + 2 * SIN (X * X)
9 LET X = X + H
10 NEXT I
11 LET F = H * (S + SIN (0) — SIN (1))/3
12 PRINT F
13 END
```

In line 11 we subtract the value sin (1) since, when carrying out cyclic computations of the sum S in lines 5-10, we already added 2 sin (1), and in the final result, the same as in formula (4), we shall have 2 sin (1) — sin (1) = sin (1). The results of computations performed according to this program will be considered in the next subsection.

4. The Error of the Formulas for Approximate Computation of Definite Integrals. The following theorem gives an estimate of the absolute error of approximation obtained by the rectangle, trapezoid, and parabola formulas of functions having a continuous fourth derivative.

Theorem 8.4.1. *Let a function $f(x)$ have a continuous fourth derivative on the closed intreval $[a, b]$. Then the following inequalities are*

valid:

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) \right| \leqslant \frac{M_2 (b-a)^3}{24n^2},$$

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2} \right| \leqslant \frac{M_2 (b-a)^3}{12n^2},$$

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{n} \sum_{i=1}^{n} \frac{f(x_{i-1})+4f\left(\frac{x_{i-1}+x_i}{2}\right)+f(x_i)}{6} \right| \leqslant \frac{M_4 (b-a)^5}{2880n^4},$$

where M_2 is the greatest value of the function $|f''(x)|$ on $[a, b]$ and M_4 the greatest value of the fourth derivative $|f^{(4)}(x)|$ on $[a, b]$.

For functions having a continuous fourth derivative the highest accuracy is possessed by the parabola formula (Simpson's formula) since its error is proportional to n^{-4}. Therefore the parabola formula is the most widely used in practice. From the given estimate for the errors of the rectangle and trapezoid formulas it follows that the first formula is approximately twice as accurate as the second one, the order of their errors being the same and proportional to n^{-2}.

Given the above programs in the BASIC language, we compute integral (6) from subsection 1 with the help of a personal computer. Its value accurate to six digits after the decimal point is equal to 0.310268.

The results of computations by the trapezoid formula are tabulated below (with six digits after the decimal point) together with the number N of subdivisions of the interval of integration.

Table 1

N	Computed value of the integral	Modulus of the absolute error
20	0.310494	0.000226
50	0.310304	0.000036
100	0.310277	0.000009
200	0.310271	0.000003

For the rectangle formula the results of computations are as follows:

Table 2

N	Computed value of the integral	Modulus of the absolute error
20	0.310156	0.000112
50	0.310250	0.000018
100	0.310264	0.000004
200	0.310267	0.000001

The results of computations by the parabola formula have a high accuracy even for small N:

Table 3

N	Computed value of the integral	Modulus of the absolute error
1	0.305181	0.005087
2	0.309944	0.000324
5	0.310260	0.000008
10	0.310268	Less than 10^{-6}

Exercises

1. Find the approximate value of the integral $\int_0^1 \frac{dx}{1+x} = \ln 2 = 0.69315$ using both the rectangle and trapezoid formulas for $n = 10$.

2. Using both the rectangle and trapezoid formulas, evaluate the integral $\int_0^4 x^2\,dx = \frac{4^3}{3} \approx 21.33$ for $n = 10$.

3. Using the rectangle, trapezoid, and parabola formulas, evaluate the integral $\int_1^e \frac{dx}{x} = 1$ dividing the interval of integration into four parts.

4. Write in the BASIC language the program for computing the integral $\int_0^1 \sin(x^2)\,dx$ making use of the formulas for left- and right-hand rectangles.

8.5. Geometric Applications of the Definite Integral

1. Computing Areas of Plane Figures. It was shown above that the definite integral of a nonnegative continuous function is equal to the area of the corresponding curvilinear trapezoid. This is the geometric meaning of the definite integral, this underlies its application to computing areas of plane figures.

As it was established earlier, the area of a curvilinear trapezoid bounded by the graph of a nonnegative continuous function $y = f(x)$, $x \in [a, b]$, the segment $[a, b]$ of the x-axis, and the line segments $x = a$ and $x = b$ $(a < b)$ is equal to

$$S = \int_a^b f(x)\,dx. \tag{1}$$

Consider a number of examples illustrating how formula (1) is used for finding areas of plane figures.

Example 1. Compute the area of the plane figure bounded by the lines $y = 3x^2 - 6x + 5$, $x = -1$, $x = 2$, and the segment $[-1, 2]$ of the axis of abscissas.

△ The given plane figure is a curvilinear trapezoid (hatched in Fig. 114), therefore its area is computed by formula (1):

$$S = \int_{-1}^{2} (3x^2 - 6x + 5)\,dx = x^3 \Big|_{-1}^{2} - 3x^2 \Big|_{-1}^{2} + 5x \Big|_{-1}^{2} = 15. \ \blacktriangle$$

Let now the function $y = f(x)$, $x \in [a, b]$, take on either negative or zero values (that is, let $f(x)$ be nonpositive continuous function). The graph of such a function is located below the axis of abscissas (Fig. 115), and

$$\int_a^b f(x)\,dx \leqslant 0.$$

Consider, further, the auxiliary function $y = -f(x)$, $x \in [a, b]$, which is a nonnegative continuous function. Consequently, the area of the curvilinear trapezoid $aA'B'b$ (Fig. 115) bounded by the graph

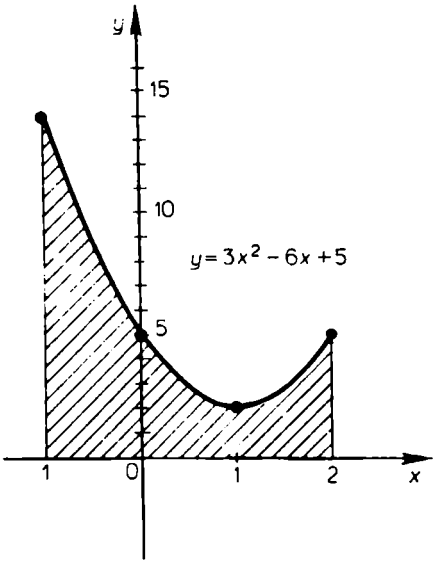

Fig. 114

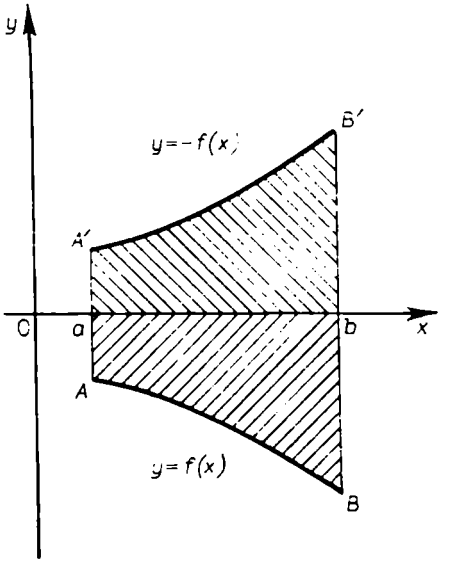

Fig. 115

of the function $y = -f(x)$, segment $[a, b]$ of the axis of abscissas, and the line segments $x = a$ and $x = b$ $(a < b)$ (Fig. 115) can be computed by formula (1), that is,

$$S = -\int_a^b f(x)\,dx. \tag{2}$$

Let us now consider the plane figure $aABb$ (Fig. 115) bounded by the graph of the function $f(x)$, segment $[a, b]$ of the axis of abscissas, and line segments $x = a$ and $x = b$. The graph of the function $y = -f(x)$ is symmetric to the graph of the function $y = f(x)$ with respect to the axis of abscissas, therefore the figures $aABb$ and $aA'B'b$ are congruent. As is known, congruent figures have equal areas, therefore the area of the plane figure $aABb$ will also be computed by formula (2).

Example 2. Compute the area of the plane figure bounded by the lines $y = \sqrt[5]{x}$, $x = -1$ and the axis of abscissas (Fig. 116).

△ The graph of the function $y = \sqrt[5]{x}$, $x \in [-1, 0]$ is situated below the x-axis, therefore, to compute the area of the given plane figure, we apply formula (2):

$$S = -\int_{-1}^{0} \sqrt[5]{x}\,dx = -\frac{5}{6}x^{\frac{6}{5}}\Big|_{-1}^{0} = \frac{5}{6}. \quad \blacktriangle$$

Let, further, $y = f(x)$, $x \in [a, b]$ be a function continuous on the closed interval $[a, b]$ whose graph intersects the interval $[a, b]$ of the axis of abscissas at a finite number of points. Making use of formulas (1) and (2), we get that the area of the plane figure bounded by the graph of the function $y = f(x)$, interval $[a, b]$ of the axis of abscissas, and the line segments $x = a$ and $x = b$ is computed by the formula

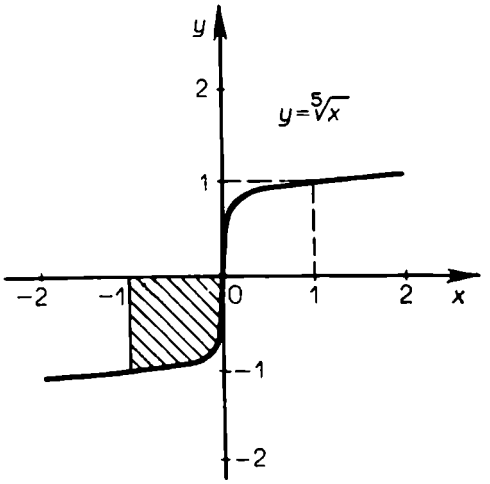

Fig. 116

$$S = \int_a^b |f(x)|\, dx. \qquad (3)$$

Example 3. Compute the area of the plane figure bounded by the graph of the function $f(x) = \sin x$, straight lines $x = -\frac{4\pi}{7}$, $x = \pi$, and the axis of abscissas (Fig. 117).

△ Solving the equation $\sin x = 0$, we get that the graph of the function $f(x) = \sin x$ on the interval $\left[-\frac{4\pi}{7}, \pi\right]$ intersects the axis

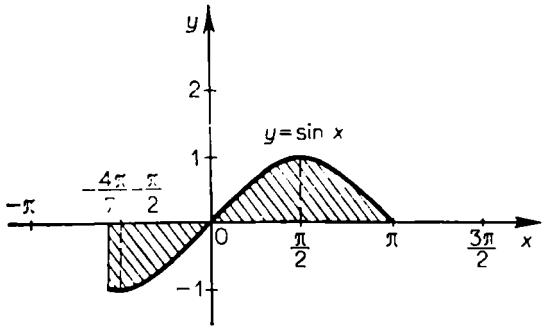

Fig. 117

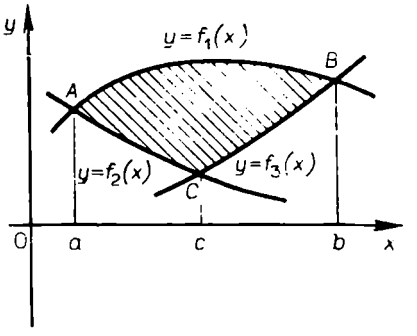

Fig. 118

of abscissas at one point $x_1 = 0$. Thus, $\sin x \leqslant 0$ on the interval $\left[-\frac{4\pi}{7}, 0\right]$ and $\sin x \geqslant 0$ on the interval $[0, \pi]$. We make use of formula (3):

$$S = \int_{-\frac{4\pi}{7}}^{\pi} |\sin x|\, dx = \int_{-\frac{4\pi}{7}}^{0} (-\sin x)\, dx + \int_0^{\pi} \sin x\, dx$$

$$= \cos x \Big|_{-\frac{4\pi}{7}}^{0} - \cos x \Big|_0^{\pi} = 3 - \cos\frac{4\pi}{7}. \quad \blacktriangle$$

If it is required to compute the area of a plane figure of a more complicated (than a curvilinear trapezoid) form, then we divide this figure into several curvilinear trapezoids. The required area is then found

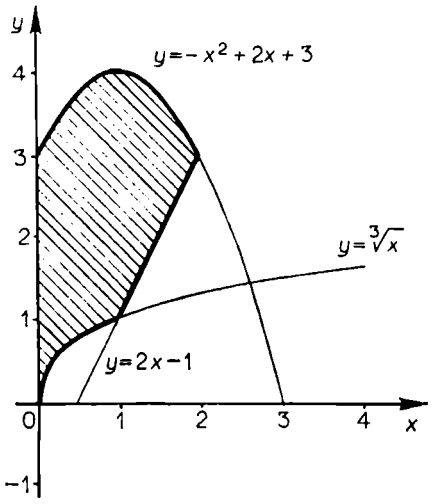

Fig. 119

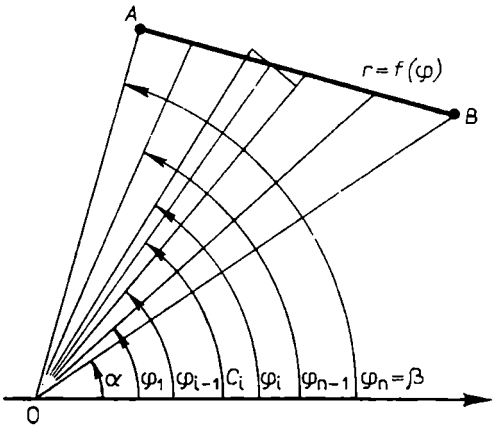

Fig. 120

as an algebraic sum of the areas of these curvilinear trapezoids. For instance, the area of the figure represented in Fig. 118 is computed by the formula

$$S=\int_a^b f_1(x)\,dx-\int_a^c f_2(x)\,dx-\int_c^b f_3(x)\,dx. \tag{4}$$

Example 4. Compute the area of the plane figure bouded by the lines $y=\sqrt[3]{x}$, $y=2x-1$, $y=-x^2+2x+3$ and the axis of ordinates (Fig. 119).

△ To find the area of the given figure (Fig. 119), we use formula (4):

$$S=\int_0^2(-x^2+2x+3)\,dx-\int_1^2(2x-1)\,dx-\int_0^1\sqrt[3]{x}\,dx$$

$$=\left(-\frac{x^3}{3}+x^2+3x\right)\Big|_0^2-(x^2-x)\Big|_1^2-\frac{3}{4}x^{\frac{4}{3}}\Big|_0^1=\frac{55}{12}. \quad \blacktriangle$$

Let us now consider the plane figure bounded by the graph of the function $r=f(\varphi)$, $\varphi\in[\alpha,\beta]$, given in the polar coordinate system and the line segments of the rays $\varphi=\alpha$ and $\varphi=\beta$ (Fig. 120). Such

a figure will be called a curvilinear sector. Its area can be computed by formula (1), however, it is more convenient to deduce a special formula for this purpose.

Let us divide the interval $[\alpha, \beta]$ by points $\varphi_i = \alpha + \frac{\beta-\alpha}{n} i$, $i = 0, 1, \ldots, n$, into n equal subintervals $[\alpha, \varphi_1]$, $[\varphi_1, \varphi_2]$, $\ldots$, $[\varphi_{n-1}, \beta]$, and choose a point $c_i \in [\varphi_{i-1}, \varphi_i]$, $i = 1, \ldots, n$ on each of them. We then form the integral sum

$$\frac{1}{2} r^2(c_1)\,\Delta\varphi_1 + \ldots + \frac{1}{2} r^2(c_n)\,\Delta\varphi_n = \frac{1}{2} \sum_{i=1}^{n} r^2(c_i)\,\Delta\varphi_i, \qquad (5)$$

where $\Delta\varphi_i = \varphi_i = \varphi_{i-1}$. Each term of sum (5) is equal to the area of the circular sector (Fig. 120), while the total sum is equal to the area of the corresponding stepped figure. Since for the continuous

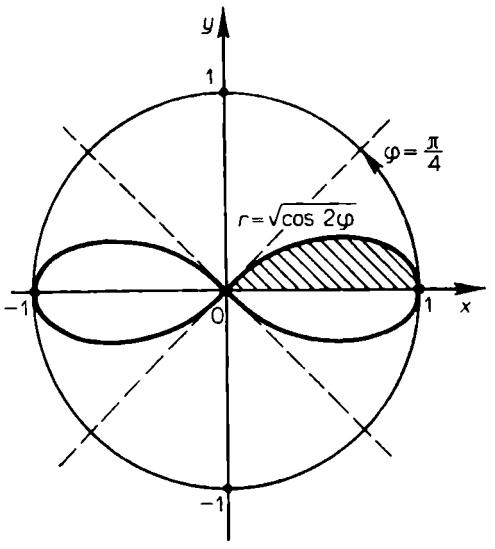

Fig. 121

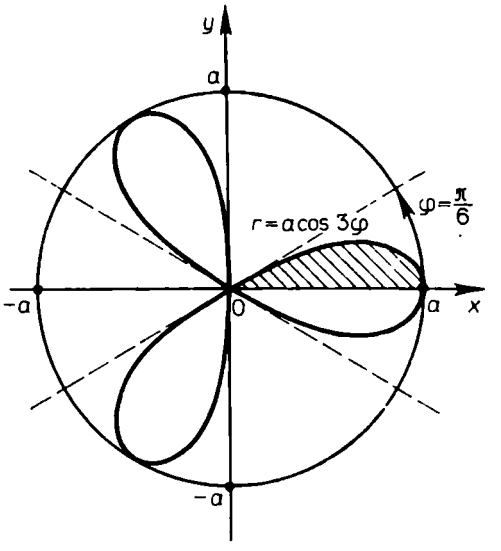

Fig. 122

function $r = f(\varphi)$, $\varphi \in [\alpha, \beta]$, the limit of integral sums (5), as $n \to \infty$, exists, we get the following formula for computing the area of the curvilinear sector S_{OAB}:

$$S_{OAB} = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\varphi)\,d\varphi.$$

Example 5. Compute the area of the figure bounded by the lemniscate $r = \sqrt{\cos 2\varphi}$.

△ The desired area is four times the area of the portion hatched in Fig. 121. Therefore

$$S = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \cos 2\varphi \, d\varphi = \int_0^{\frac{\pi}{4}} \cos 2\varphi d(2\varphi)$$

$$= \int_0^{\frac{\pi}{2}} \cos u \, du = \sin u \Big|_0^{\frac{\pi}{2}} = 1. \ \blacktriangle$$

Example 6. Compute the area of the "three-petalled rose" $r = a \cos 3\varphi$.

△ The required area is six times the portion hatched in Fig. 122. Therefore

$$S = 6 \cdot \frac{1}{2} \int_0^{\frac{\pi}{6}} a^2 \cos^2 3\varphi \, d\varphi = a^2 \int_0^{\frac{\pi}{6}} \cos^2 3\varphi d(3\varphi)$$

$$= a^2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2u}{2} \, du = \frac{\pi a^2}{4}. \ \blacktriangle$$

2. Computing the Arc Length of a Plane Curve. Let there be given a plane curve AB (Fig. 123) whose equation is $y = f(x)$, $x \in [a, b]$, where $f(x)$ is a continuously differentiable function on the closed interval $[a, b]$. Let us divide the interval $[a, b]$ into n equal parts with the help of the points $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, \ldots, n$. Through the points x_i we draw straight lines parallel to the y-axis and denote by M_i the points of intersection of these lines with the curve AB. Joining these points with chords, we get a polygonal line $AM_1M_2 \ldots M_{n-1}B$ inscribed into the curve AB. Let the perimeter of this polygonal line be equal to P_n. The arc length AB is the number l equal to the limit of the sequence of the perimeters $\{P_n\}$:

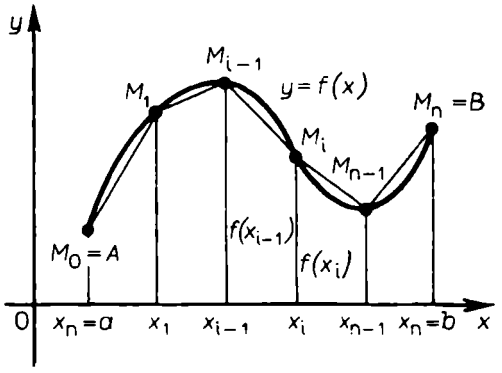

Fig. 123

$$l = \lim_{n \to \infty} P_n.$$

Let us now derive a formula for computing the arc length of AB. To this end, we first compute the perimeter of the polygonal line $AM_1M_2 \ldots M_{n-1}B$. The point M_{i-1} with coordinates x_{i-1} and $f(x_{i-1})$ and the point M_i with coordinates x_i and $f(x_i)$ are endpoints of the ith segment of the polygonal line. We then compute the length of the ith segment using the formula for the distance between two points in the plane:

$$l_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}. \tag{1}$$

Since $f(x)$ is a continuously differentiable function on the interval $[a, b]$, by Lagrange's formula, we have

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}), \tag{2}$$

where c_i is some point belonging to the open interval (x_{i-1}, x_i). Substituting expression (2) into formula (1), we get

$$l_i = \sqrt{1 + (f'(c_i))^2}\,\Delta x_i, \tag{3}$$

where $\Delta x_i = x_i - x_{i-1}$. Consequently, the perimeter of the polygonal line $AM_1M_2 \ldots M_{n-1}B$ is equal to

$$P_n = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\,\Delta x_i.$$

Thus, we have obtained an integral sum for the continuous function $\sqrt{1 + (f'(x))^2}$ on the interval $[a, b]$. Since the limit of this sum exists, as $n \to \infty$, we shall have, by definition,

$$l = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\,\Delta x_i = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Thus,

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\,dx. \tag{4}$$

Example. Compute the length of the arc of the parabola $y = \frac{1}{2}x^2$ from its vertex $A(0, 0)$ to the point $B\left(1, \frac{1}{2}\right)$.

△ To compute the arc length, we make use of formula (4). Since $y' = x$, we have $l = \int_0^1 \sqrt{1 + x^2}\,dx$. It was shown above (see Example 4 in subsection 2 of 7.3) that

$$\int \sqrt{1 + x^2}\,dx = \frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\ln\left|x + \sqrt{1 + x^2}\right| + C.$$

Consequently,

$$l=\int_0^1 \sqrt{1+x^2}\,dx=\frac{1}{2}\left[x\sqrt{1+x^2}+\ln\left(x+\sqrt{1+x^2}\right)\right]\Big|_0^1$$

$$=\frac{\sqrt{2}+\ln(1+\sqrt{2})}{2}. \blacktriangle$$

3. Computing Volumes of Solids of Revolution. Consider the solid generated by rotating about the x-axis the curvilinear trapezoid $aABb$ bounded by the graph of a nonnegative continuous function

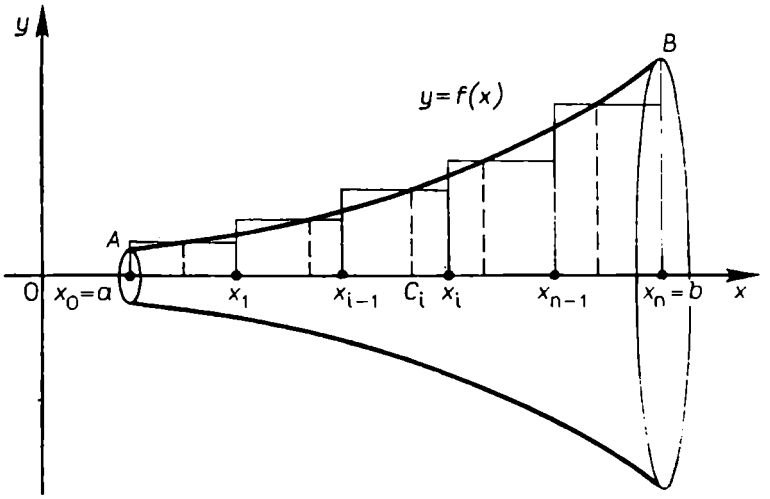

Fig. 124

$y = f(x)$, $x \in [a, b]$, the axis of abscissas, and the line segments of the straight lines $x = a$, and $x = b$ (Fig. 124). With the help of the points $x_i = a + \frac{b-a}{n}\,i$, $i = 0, 1, \ldots, n$, we divide the interval $[a, b]$ into n equal parts. On each of the subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, we choose a point $c_i \in [x_{i-1}, x]$. We then form the integral sum

$$\pi f^2(c_1)\,\Delta x_1+\ldots+\pi f^2(c_n)\,\Delta x_n=\pi\sum_{i=1}^{n} f^2(c_i)\,\Delta x_i, \qquad (1)$$

where $\Delta x_i = x_i - x_{i-1}$. Each term of sum (1) is equal to the volume of a circular cylinder, while the total sum yields the volume of the corresponding stepped solid. For continuous functions $f(x)$, $x \in [a, b]$ the limit of integral sums (1), as $n \to \infty$, exists and is equal to the volume V of the solid of revolution under consideration

$$V=\pi\lim_{n\to\infty}\sum_{i=1}^{n} f^2(c_i)\,\Delta x_i=\pi\int_a^b f^2(x)\,dx. \qquad (2)$$

Example. Compute the volume of the solid generated by rotating the parabola $y = x^2$ about the axis of abscissas on the section from $x = 0$ to $x = b > 0$.

△ We compute by formula (2):

$$V = \pi \int_a^b x^4\,dx = \frac{\pi x^5}{5}\Big|_0^b = \frac{\pi b^5}{5}. \quad \blacktriangle$$

4. Computing the Surface Area of a Solid of Revolution. Figure 124 represents a solid of revolution. It is required to compute the area of the surface generated by rotating the curve AB about the axis of abscissas. Let the function $y = f(x)$, $x \in [a, b]$, be continuously differentiable on the closed interval $[a, b]$. Through the points $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, \ldots, n$, we draw straight lines parallel to the y-axis and denote the points of intersection of these lines and the curve AB by M_i. Connecting these points by chords, we get a polygonal line $AM_1M_2 \ldots M_{n-1}B$. When rotated about the axis of abscissas, this polygonal line generates a surface consisting of the lateral surfaces of the truncated cones generated by the rotating segments of the polygonal line $AM_1M_2 \ldots M_{n-1}B$. Let the area of this surface be equal to S_n. The surface area of the solid of revolution is defined as the number S equal to the limit of the sequence of areas $\{S_n\}$:

$$S = \lim_{n\to\infty} S_n.$$

The area of the surface bounded by the polygonal line $AM_1M_2 \ldots M_{n-1}B$ is equal to:

$$S_n = \sum_{i=1}^{n} 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} l_i$$

$$= \pi \sum_{i=1}^{n} [f(x_{i-1}) + f(x_i)] \sqrt{1 + (f'(c_i))^2}\,\Delta x_i. \qquad (1)$$

where we have used formula (3) from subsection 2. Sum (1) is not an integral sum for the function

$$2\pi f(x) \sqrt{1 + (f'(x))^2}, \qquad (2)$$

since the term corresponding to the interval $[x_{i-1}, x_i]$ contains several points of this subinterval, namely: x_{i-1}, x_i, c_i. But we can prove that the limit of sum (1) is equal to the limit of the integral sum for func-

tion (2), that is,

$$S=\lim_{n\to\infty} S_n=\pi \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_{i-1})+f(x_i)]\sqrt{1+(f'(c_i))^2}\,\Delta x_i$$

$$=2\pi \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\sqrt{1+(f'(c_i))^2}\,\Delta x_i$$

$$=2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Thus,

$$S=2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx. \tag{3}$$

Example. Compute the surface area of the paraboloid generated by rotating the parabola arc $y=\sqrt{x}$, $x\in[0, 2]$, about the axis of abscissas.

△ To compute the required surface area, we make use of formula (3). Since $f'(x)=\frac{1}{2\sqrt{x}}$, we have

$$S=2\pi \int_0^2 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx=\pi \int_0^2 \sqrt{1+4x}\,dx$$

$$=\frac{\pi}{4}\int_1^9 \sqrt{u}\,du=\frac{\pi}{6}u^{\frac{3}{2}}\Big|_1^9=\frac{13\pi}{3}. \ \blacktriangle$$

Exercises

1. Compute the area of the plane figure bounded by the indicated lines:

(a) $y=6x-x^2$, $y=0$;

(b) $y=x^2+1$, $y=0$, $x=0$, $x=2$;

(c) $y=8+2x-x^2$, $y=2x+4$;

(d) $y=x^3-4x$, $y=0$;

(e) $y=2x+16$, $y=8-7x-x^2$, $x=0$;

(f) $y^2-4x=0$, $x-y=0$;

(g) $y^2=ax$, $x^2=by$;

(h) $y^2=2x$, $2y=x^2$;

(i) $y=\ln x$, $y=0$, $x=2$, $x=8$;

(j) $y=\sin x$, $y=\frac{2x}{\pi}$;

(k) $y=x^3$, $x+y=2$, $y=0$;

(l) $y=\tan x$, $x=\frac{\pi}{3}$, $y=0$;

(m) $y=\arcsin x$, $x=\frac{1}{2}$, $x=0$.

2. Compute the area of the plane figure bounded by the indicated lines:

(a) $r = a(1 - \cos\varphi)$; (b) $r = a\cos 2\varphi$; (c) $r = a\sin 3\varphi$;

(d) $r = a\sin 2\varphi$; (e) $r = a(\sin\varphi + \cos\varphi)$;

(f) $r = \frac{a}{\varphi}$, $\frac{\pi}{4} \leqslant \varphi \leqslant 2\pi$.

3. Compute the arc lengths of the following curves:

(a) $y = \sqrt{x}$, $x \in [0, 1]$; (b) $y = \ln x$, $x \in [\sqrt{3}, \sqrt{8}]$; (c) $y = e^x$, $x \in [0, 1]$; (d) $y = \ln\cos x$, $x \in [0, \alpha]$, where $\alpha < \frac{\pi}{2}$; (e) $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between the points whose abscissas are equal to 0 and a; (f) $y = \ln\sin x$ between the points whose abscissas are equal to $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

4. Compute the volumes of the following solids of revolution about the axis of abscissas:

(a) $xy = 4$, $x = 1$, $x = 4$, $y = 0$; (b) $(y - a)^2 = ax$, $x = 0$, $y = 2a$; (c) $y = \sin x$ (one half-wave), $y = 0$; (d) $y = \frac{1}{1+x^2}$, $x = \pm 1$, $y = 0$; (e) $y = x^3$, $x = 0$, $y = 8$.

5. Compute the area of the surface generated by rotating the given curve about the axis of abscissas:

(a) $x^2 + y^2 = R^2$; (b) $y = \sin x$ (one half-wave); (c) the arc of the curve $y = \frac{x^3}{3}$ from $x = -2$ to $x = 2$; (d) the curve $y^2 = 4 + x$ cut off by the straight line $x = 2$.

8.6. The Definite Integral in Physical Problems

1. The Work Done by a Variable Force. It was shown in 8.1 that the work A of a variable force $F(x)$ on the interval $[a, b]$ is

$$A = \int_a^b F(x)\,dx. \tag{1}$$

Example. Compute the minimal work done by an external force which is necessary to displace a body of mass m from the Earth's surface to a point in outer space located at a distance R from the Earth's centre (Fig. 125).

△ The body of mass m is acted upon by the gravitational force F_{gr} toward the centre of the Earth. This force depends on the distance r between the body and the centre of the Earth:

$$F_{gr} = \frac{GM_E m}{r^2}, \tag{2}$$

where M_E is the mass of the Earth, and G gravitational constant. To remove the body from the Earth's surface, it is required to apply

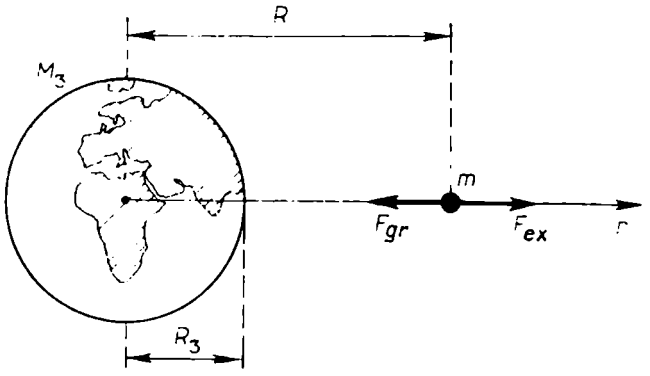

Fig. 125

to it an external force F_{ex} in the opposite direction, the magnitude of this force being greater than, or equal to, F_{gr}. Minimal work will be done in the limiting case: $F_{ex}(r) = F_{gr}(x)$. The desired work is computed by formula (1):

$$A_{min} = \int_{R_E}^{R} F_{ex}(r)\,dr = \int_{R_E}^{R} F_{gr}(r)\,dr = GM_E m \int_{R_E}^{R} \frac{dr}{r^2}$$

$$= -\frac{GM_E m}{r}\Bigg|_{R_E}^{R} = GM_E m\left(\frac{1}{R_E} - \frac{1}{R}\right).$$

Thus,

$$A_{min}(R) = GM_E m\left(\frac{1}{R_E} - \frac{1}{R}\right). \tag{3}$$

From formula (3) we find that to displace the body from the Earth's surface to an infinitely remote point of outer space ($R \to \infty$), it is necessary to perform finite work

$$A_{min}(\infty) = \frac{GM_E m}{R_E}.$$

To make the above calculations as simple as possible, we have neglected the attractive forces acting on the body of mass m due to other space objects. ▲

2. Computing Displacement and Path Length. If we are given the equation of rectilinear motion (along the x-axis) performed by a material point

$$x = x(t),\ t \in [t_0, t_f],$$

where x is the coordinate of the material point on the x-axis, t the time of motion, and t_0 and t_f the initial and final instants of observation, then the displacement P of the point during the time interval from $t = t_1 \in [t_0, t_f]$ to $t = t_2 \in [t_0, t_f]$ is determined by the formula

$$P = x(t_2) - x(t_1).$$

Let there be given the dependence of the velocity of motion of the material point on time:

$$v = \varphi(t),\ t \in [t_0, t_f].$$

It is required to compute the displacement P. According to the definition of the velocity of rectilinear motion (see Chapter 6), we have:

$$v = \varphi(t) = \frac{dx(t)}{dt},\ t \in [t_0, t_f].$$

Hence we find the desired displacement:

$$P = x(t_2) - x(t_1) = \int_{t_1}^{t_2} \varphi(t)\,dt. \tag{1}$$

If the motion is performed all the time in the same direction ($\varphi(t) \geqslant 0$, $t \in [t_0, t_f]$), then the path L covered by the material point during the time from $t = t_1$ to $t = t_2$ coincides with the displacement P. In the general case, when nonuniform motion is performed along the x-axis in opposite directions, the path L is computed by the formula

$$L = \int_{t_1}^{t_2} |\varphi(t)|\,dt.$$

Example. A stone is thrown vertically upward from the Earth's surface with an initial velocity v_0. Neglecting air resistance, compute: (a) the height of climb H_{cl}; (b) the time of climb t_{cl}; (c) the total time of motion of the stone before it fell to the Earth T; (d) the resultant displacement of the stone P; (e) the total path L travelled by the stone.

△ It is known from physics that near the surface of the Earth (if air resistance is neglected) a body performs uniformly accelerated motion in which velocity changes with time according to the law

$$v(t) = v_0 - gt, \tag{2}$$

where g is free fall acceleration, the instant of time $t = 0$ coincides with that of throwing the stone, and the x-axis is directed vertically upward. The stone moves upward as long as $v(t) \geqslant 0$, so that at the extreme upper point we have $v(t_{cl}) = 0$. From formula (2) we find the climbing time t_{cl}:

$$t_{cl} = \frac{v_0}{g}.$$

The height of climb H_{cl} is equal to the displacement during the time interval $[0, t_{cl}]$:

$$H_{cl} = \int_0^{t_{cl}} v(t)\,dt = \int_0^{t_{cl}} (v_0 - gt)\,dt = \left(v_0 t - \frac{gt^2}{2}\right)\Big|_0^{t_{cl}}$$

$$= v_0 t_{cl} - \frac{g t_{cl}^2}{2} = \frac{v_0^2}{2g}.$$

Since the total path covered during the time interval from $t = 0$ till $t = T$ is equal to $2H_{cl}$, we have

$$L = 2H_{cl} = \frac{v_0^2}{g} = \int_0^T |v(t)|\,dt = \int_0^{t_{cl}} (v_0 - gt)\,dt + \int_{t_{cl}}^T (gt - v_0)\,dt$$

$$= 2\left(v_0 t_{cl} - \frac{g t_{cl}^2}{2}\right) + \frac{gT^2}{2} - v_0 T = \frac{v_0^2}{g} + \frac{gT^2}{2} - v_0 T,$$

whence we easily find the total time during which the stone was in motion (since $T \neq 0$):

$$T = \frac{2v_0}{g} = 2t_{cl},$$

that is, the time of climb is equal to the time of fall. The resultant displacement is computed by formula (1):

$$P = \int_0^T (v_0 - gt)\,dt = v_0 T - \frac{gT^2}{2} = \frac{2v_0^2}{g} - \frac{2v_0^2}{g} = 0,$$

which is quite obvious: the stone returned to the initial point of its motion. ▲

As an example of plane nonrectilinear motion, let us consider the motion of a fixed point A of the rim of a wheel rolling without slippage whose radius is R (Fig. 126). It is required to compute the length of the path travelled by the point A as the wheel completed one turn. Suppose that at the initial instant of time $t = 0$ the point A occupied the lowest position coinciding with the origin of the xOy-system of coordinates. The dependence of the coordinates of the

point A on the angle of rotation ψ has the form:

$$\begin{aligned} y &= R - R\cos\psi, \\ x &= R\psi - R\sin\psi, \quad \psi \in [0,\ 2\pi]. \end{aligned} \tag{3}$$

Formulas (4) represent the function $y(x)$ parametrically, where the angle of rotation ψ serves as a parameter. According to formu-

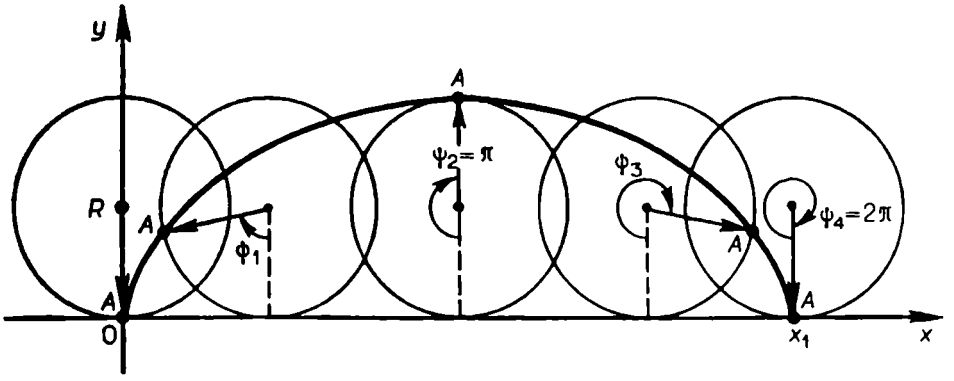

Fig. 126

la (4) from subsection 2 of 8.5, the path L travelled by the point A is equal to (the arc length of the curve $y(x)$):

$$L = \int_0^{x_1} \sqrt{1 + (y'(x))^2}\, dx. \tag{4}$$

For the function $y(x)$ represented parametrically, we have

$$dx = \frac{dx}{d\psi} \cdot d\psi, \quad y'(x) = \frac{dy}{dx} = \frac{\dfrac{dy}{d\psi}}{\dfrac{dx}{d\psi}},$$

and in this case formula (4) takes the form:

$$L = \int_0^{2\pi} \sqrt{(x'(\psi))^2 + (y'(\psi))^2}\, d\psi. \tag{5}$$

Finding the derivatives

$$x'(\psi) = R - R\cos\psi, \quad y'(\psi) = R\sin\psi,$$

we evaluate integral (5):

$$L=\int_0^{2\pi}\sqrt{2R^2(1-\cos\psi)}\,d\psi=\int_0^{2\pi}\sqrt{4R^2\sin^2\frac{\psi}{2}}\,d\psi$$

$$=2R\int_0^{2\pi}\sin\frac{\psi}{2}\,d\psi$$

$$=4R\int_0^{2\pi}\sin\frac{\psi}{2}\,d\left(\frac{\psi}{2}\right)\quad=4R\int_0^{\pi}\sin u\,du=-4R\cos u\Big|_0^{\pi}=8R.$$

3. Moments of Inertia of Material Lines and Plates. If a material point performs uniform circular motion, then its linear v and angular ω velocities are related as follows:

$$v=\omega r, \tag{1}$$

where r is the radius of the circle the point moves in. If k material points with masses $m_1, m_2, \ldots, m_k$ rotate about the same axis with an equal angular velocity ω but in circles of different radii $r_1, r_2, \ldots, r_k$, then their total kinetic energy E ((1) being taken into account) is equal to

$$E=\frac{m_1v_1^2}{2}+\ldots+\frac{m_kv_k^2}{2}=\frac{m_1\omega^2r_1^2}{2}+\ldots+\frac{m_k\omega^2r_k^2}{2}. \tag{2}$$

If for a system of material points we introduce the quantity I characterizing their rotary motion

$$I=m_1r_1^2+m_2r_2^2+\ldots+m_kr_k^2=\sum_{i=1}^{k}m_ir_i^2,$$

then the expression for the total kinetic energy of a system will take a simple form:

$$E=\frac{I\omega^2}{2}.$$

The quantity I is called the moment of inertia of a system of material points with respect to a given axis of rotation.

Let now a plane material body whose mass M is uniformly distributed over the area of the curvilinear trapezoid $aABb$ (Fig. 127) rotate with an angular velocity ω about the y-axis. Let us compute the moment of inertia I_y of the curvilinear trapezoid $aABb$ (more exactly, of the plane material body having the shape of a curvilinear trapezoid) with respect to the y-axis, that is, the proportionality factor in the formula

$$E_{aABb}=I_y\frac{\omega^2}{2}, \tag{3}$$

where E_{aABb} is the kinetic energy of the curvilinear trapezoid $aABb$. For a discrete system of material points kinetic energy is computed by formula (2). Let us determine the kinetic energy for a distributed material body that is represented by the material

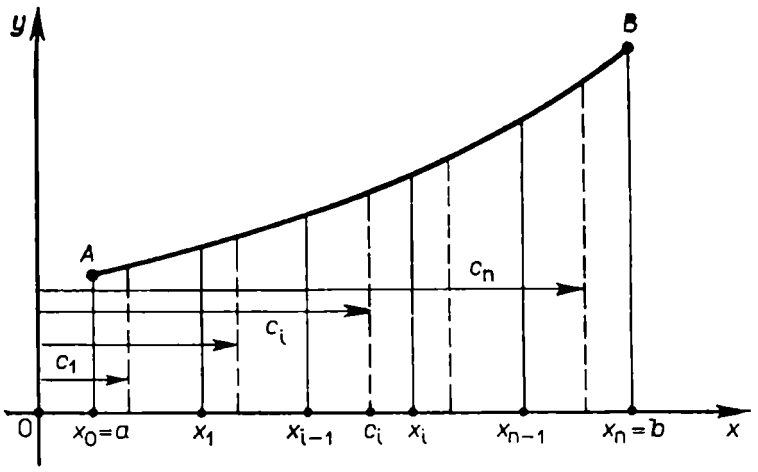

Fig. 127

curvilinear trapezoid in the following way. We divide the closed interval $[a, b]$ by the points $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, \ldots, n$, into n equal parts. On each of the subintervals thus obtained we choose a point $c_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$. We then form the sum

$$\frac{\rho\omega^2}{2} f(c_1) c_1^2 \Delta x_1 + \ldots + \frac{\rho\omega^2}{2} f(c_n) c_n^2 \Delta x_n = \frac{\rho\omega^2}{2} \sum_{i=1}^{n} f(c_i) c_i^2 \Delta x_i, \qquad (4)$$

where ρ is the density of the curvilinear trapezoid, that is, the mass per unit area:

$$\rho = \frac{M}{S_{aABb}} = \frac{M}{\int_a^b f(x)\, dx}. \qquad (5)$$

For sufficiently narrow strips into which the curvilinear trapezoid is partitioned by the vertical lines passing through the points x_i (Fig. 127), the terms of sum (4) are approximately equal to the kinetic energy of rotary motion of separate strips. Indeed, the mass of the ith strip is approximately equal to $\rho f(c_i)\, \Delta x_i$, and the linear velocity of different points of the ith strip is approximately equal to ωc_i (see formula (1)). Thus, the sum (4) is approximately equal to the kinetic energy of the entire curvilinear trapezoid which is

defined as the limit of sum (4) as $n \to \infty$:

$$E_{aABb} = \frac{\rho\omega^2}{2} \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\, c_i^2 \Delta x_i. \tag{6}$$

Since the last sum is an integral sum for the function $f(x)\, x^2$ on the interval $[a, b]$, we have

$$E_{aABb} = \frac{\rho\omega^2}{2} \int_a^b x^2 f(x)\, dx. \tag{7}$$

From formulas (3), (5), and (7) we finally find:

$$I_y = \rho \int_a^b x^2 f(x)\, dx = M \frac{\int_a^b x^2 f(x)\, dx}{\int_a^b f(x)\, dx}. \tag{8}$$

Example 1. Compute the moment of inertia with respect to the y-axis of a homogeneous plate of mass M having the shape of a curvilinear trapezoid bounded by the lines $y = x^2$, $y = 0$, $x = 0$, $x = a$ $(a > 0)$.

△ The desired moment of inertia is computed by formula (8):

$$\int_0^a x^4\, dx = \frac{a^5}{5}, \quad \int_0^a x^2\, dx = \frac{a^3}{3},$$

and, consequently,

$$I_y = M \frac{\int_0^a x^4\, dx}{\int_0^a x^2\, dx} = \frac{3Ma^2}{5}. \quad \blacktriangle$$

Let now the mass M be uniformly distributed on the line $y = f(x)$, $x \in [a, b]$ which in this case is called a material line, and the line density, that is, the mass of the unit length of the line be equal to ρ. We can show that the moment of inertia of a material line with respect to the y-axis is equal to

$$I_y = \rho \int_a^b x^2 \sqrt{1 + (f'(x))^2}\, dx. \tag{9}$$

Example 2. Compute the moment of inertia of a thin homogeneous bar of length l with respect to its endpoint.

△ We bring the bar into coincidence with the segment $0 \leqslant x \leqslant l$ of the x-axis. The moment of inertia of the bar with respect to its endpoint, that is, the moment of inertia with respect to any axis perpendicular to the bar and passing through its endpoint will be equal to the moment of inertia with respect to the y-axis. Setting in formula (9) $f(x) = 0$, $x \in [0, l]$, that is, $f'(x) = 0$, $x \in [0, l]$, we have

$$I_y = \rho \int_0^l x^2\, dx = \rho \frac{x^3}{3}\Big|_0^l = \rho \frac{l^3}{3}.$$

If the mass M of the bar is given, then $\rho = \frac{M}{l}$, and

$$I_y = \frac{Ml^2}{3}. \ ▲$$

Example 3. Compute the moment of inertia of a thin homogeneous bar of length l with respect to its midpoint.

△ We bring the bar into coincidence with the segment $-\frac{l}{2} \leqslant x \leqslant \frac{l}{2}$ of the x-axis. The moment of inertia of the bar with respect to its midpoint, that is, the moment of inertia with respect to any axis perpendicular to the bar and passing through its midpoint will be equal to the moment of inertia with respect to the y-axis. Setting in formula (9) $f(x) = 0$, $x \in \left[-\frac{l}{2}, \frac{l}{2}\right]$, that is, $f'(x) = 0$, $x \in \left[-\frac{l}{2}, \frac{l}{2}\right]$, we have

$$I_y = \rho \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\, dx = \rho \frac{x^3}{3}\Bigg|_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{\rho l^3}{12}.$$

Since $\rho = \frac{M}{l}$, where M is the mass of the bar, we have

$$I_y = \frac{Ml^2}{12},$$

that is, to one-fourth of the moment of inertia of the bar with respect to its endpoint. ▲

4. Coordinates of the Centre of Masses of a Plane Curve. Let there be given an xOy-system of rectangular Cartesian coordinates in the plane. Recall that the centre of masses of a discrete system of material points is defined as a point C in the plane with coordinates

x_C, y_C which are computed by the formulas

$$x_C = \frac{1}{M} \sum_{i=1}^{k} m_i x_i, \quad y_C = \frac{1}{M} \sum_{i=1}^{k} m_i y_i, \tag{1}$$

where $M = \sum_{i=1}^{k} m_i$ is the total mass of a system of k material points; m_i, x_i, and y_i masses and coordinates of individual points of the system, respectively. Whenever material points move, the centre of masses also moves, and if the system is not acted upon by external forces, then the centre of masses performs uniform rectilinear motion (or rests, in particular). The important part played by the centre of masses in mechanics is explained by that to describe the motion of a system in general, it is convenient to relate the origin with the centre of masses, that is, to place the origin of the new coordinate system at a point in the plane whose coordinates in the old coordinates are computed by formula (1).

In the case of a material line or plane figure, the coordinates of the centre of masses are found with the help of the definite integral.

Let a curve AB of length l be the graph of a function $y = f(x)$, $x \in [a, b]$, having a continuous derivative. We shall assume that the curve AB is homogeneous, that is, the line density ρ of mass distribution is constant. We divide the curve into parts of length $\Delta l_i \approx \sqrt{1 + (f'(x_i))^2}\, \Delta x_i$. The mass of each of these parts is equal to $\rho \Delta l_i$. Taking these parts for material points lying at a distance $y_i = f(x_i)$ from the x-axis and at a distance x_i from the y-axis, we get approximately

$$x_C \approx \frac{\rho}{M} \sum_{i=1}^{n} x_i \sqrt{1 + (f'(x_i))^2}\, \Delta x_i,$$

$$y_C \approx \frac{\rho}{M} \sum_{i=1}^{n} f(x_i) \sqrt{1 + (f'(x_i))^2}\, \Delta x_i, \tag{2}$$

where M is the mass of the entire curve:

$$M = \rho l = \rho \int_a^b \sqrt{1 + (f'(x))^2}\, dx. \tag{3}$$

Here we have used formula (4) from 8.5 (subsection 2) for the length l of the curve AB. Since formula (2) contains the integral sums for the continuous functions $x\sqrt{1 + (f'(x))^2}$ and $f(x)\sqrt{1 + (f'(x))^2}$, passing to the limit in formula (2), as $n \to \infty$, and taking into

account formula (3), we get

$$x_C = \frac{\int_a^b x\sqrt{1+(f'(x))^2}\,dx}{\int_a^b \sqrt{1+(f'(x))^2}\,dx}, \quad y_C = \frac{\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx}{\int_a^b \sqrt{1+(f'(x))^2}\,dx}. \qquad (4)$$

Note. Formula (4) implies the equality

$$2\pi y_C l = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

According to formula (3) from 8.5 subsection 4), the right-hand side of the given equality contains the expression for the area S of the surface of revolution so that

$$S = 2\pi y_C l,$$

and the area of the surface generated by rotating the curve about an axis in its plane, but not intersecting it, is equal to the arc length multiplied by the circumference of the circle described by the centre of masses of the arc during this rotation. This statement is called Guldin's first theorem.

Example 1. Compute the coordinates of the centre of masses of the semicircle

$$y = \sqrt{R^2-x^2}, \quad -R \leqslant x \leqslant R.$$

△ Since $y' = -\frac{x}{\sqrt{R^2-x^2}}$, we have $\sqrt{1+(y')^2} = \frac{R}{\sqrt{R^2-x^2}}$. The length of the semicircle is equal to πR, therefore from formulas (4) we find:

$$x_C = \frac{1}{\pi R}\int_{-R}^{R} x\frac{R\,dx}{\sqrt{R^2-x^2}} = -\frac{1}{\pi}\int_{-R}^{R}\frac{d(R^2-x^2)}{2\sqrt{R^2-x^2}}$$

$$= -\frac{1}{\pi}\sqrt{R^2-x^2}\Big|_{-R}^{R} = 0,$$

$$y_C = \frac{1}{\pi R}\int_{-R}^{R}\sqrt{R^2-x^2}\,\frac{R}{\sqrt{R^2-x^2}}\,dx = \frac{2R}{\pi}. \quad ▲$$

Example 2. Find the coordinates of the centre of masses of the arc of the circle $x^2 + y^2 = R^2$ located in the first quadrant.

△ The given figure is symmetric about the angle bisector of the first quadrant, therefore the ordinate and abscissa of the centre of masses of this figure are equal to each other. To find the ordinate

of the centre of masses, we make use of Guldin's first theorem. It is easy to see that, when rotated about an axis, a quarter of a circle generates a hemisphere, therefore its area will be equal to $2\pi R^2$. Since the arc length of a quarter of a circle is equal to $\frac{\pi R}{2}$, according to the theorem, we get

$2\pi R^2 = 2\pi y_C \cdot \frac{\pi R}{2}$, that is, $y_C = \frac{2R}{\pi}$. Thus, $C\left(\frac{2R}{\pi}, \frac{2R}{\pi}\right)$. ▲

5. Coordinates of the Centre of Masses of a Plane Figure. Let there be given a curvilinear trapezoid bounded by the graph of a nonnegative function $y = f(x)$, $x \in [a, b]$, the x-axis, and straight lines $x = a$ and $x = b$, the mass M being distributed continuously and uniformly over the trapezoid. In order to compute the coordinates of the centre of masses of this figure, we partition it into n parts (Fig. 127) by straight lines parallel to the axis of ordinates and passing through the points $x_i = a + \frac{b-a}{n} i$, $i = 0, 1, \ldots, n$. The mass of one such part $\Delta m_i \approx \rho f(c_i)\, \Delta x_i$. Taking these parts for material points lying at a distance $y_i = \frac{1}{2} f(c_i)$ from the x-axis and at a distance c_i from the y-axis, we get approximately

$$x_C \approx \frac{\rho}{M} \sum_{i=1}^{n} c_i f(c_i)\, \Delta x_i,\ y_C \approx \frac{\rho}{M} \sum_{i=1}^{n} \frac{1}{2} f^2(c_i)\, \Delta x_i, \tag{5}$$

where the mass M and density ρ are related as follows:

$$M = \rho S_{aABb} = \rho \int_a^b f(x)\, dx. \tag{6}$$

Since formula (5) contains the integral sums for the functions $xf(x)$ and $\frac{1}{2} f^2(x)$, then, passing to the limit, as $n \to \infty$, in formula (5) and taking into account equalities (6), we have:

$$x_C = \frac{\int_a^b xf(x)\, dx}{\int_a^b f(x)\, dx},\quad y_C = \frac{1}{2} \frac{\int_a^b f^2(x)\, dx}{\int_a^b f(x)\, dx}. \tag{7}$$

Multiplying the last equality by $2\pi S_{aABb} = 2\pi \int_a^b f(x)\, dx$, we get

$$2\pi S_{aABb}\, y_C = \pi \int_a^b f^2(x)\, dx. \tag{8}$$

According to formula (2) from 8.5 (subsection 3), the right-hand side of formula (8) contains the volume V of a solid of revolution. Thus,

$$V = S_{aABb} \cdot 2\pi y_C,$$

that is, the volume of a solid of revolution generated by a curvilinear trapezoid rotating about the axis of abscissas is equal to the area of this trapezoid multiplied by the circumference of the circle described by the centre of its masses. This is the Guldin second theorem.

Example Find the coordinates of the centre of masses of the semicircle $x^2 + y^2 \leqslant R^2$, $y \geqslant 0$.

△ To find the ordinate, we make use of Guldin's second theorem: $y_C = \frac{V}{2\pi S}$. Since the volume of the sphere $V = \frac{4}{3}\pi R^3$, and the area of the semicircle $S = \frac{\pi R^2}{2}$, we have:

$$y_C = \frac{\frac{4}{3}\pi R^3}{2\pi \cdot \frac{\pi R^2}{2}} = \frac{4R}{3\pi}.$$

Since the figure is symmetric about the axis of ordinates, it follows that $x_C = 0$. Thus, $C\left(0, \frac{4R}{3\pi}\right)$. ▲

Exercises

1. A cube-shaped vessel is filled up with water. Derive the formula for the force of the hydrostatic pressure of the water on the lateral wall of the vessel.
2. Situated on the x-axis are a material segment $[0, l]$ and a material point $A(a, 0)$ whose masses are M and m, respectively. Derive the formula for the force of their mutual gravitational attraction.
3. Compute the minimal work to be done for compressing a spring by $\Delta l_1 = 5$ cm if it is known that the force $F = 2N$ compresses this spring by $\Delta l_2 = 1$ cm.
4. The space under the piston in a cylinder is filled by a gas which satisfies the Mendeleev-Clapeyron law. The initial volume and pressure are equal to V_0 and p_0. Compute the work done by isothermic compression of the gas to the volume V_1 ($V_1 < V_0$).
5. A motorcycle performs a rectilinear motion. Its speed during the time interval $[0, T]$ varies according to the law $v(t) = At - Bt^2$, $A > 0$, $B > 0$, $t \in [0, T]$. Compute the displacement of the motorcycle during the indicated time interval and the path covered.
6. Compute the moment of inertia with respect to the y-axis of the triangular plate of mass M bounded by the lines $y = 2x$, $x = 3$, and $y = 0$.

7. Compute the kinetic energy of a rectangular plate $a \leqslant x \leqslant l_1$, $0 \leqslant y \leqslant l_2$ rotating with an angular velocity ω about the y-axis if the mass of the plate is M.

8. Compute the moment of inertia with respect to the y-axis of the circle $x^2 + y^2 \leqslant R^2$ whose mass is M.

9. Compute the moment of inertia with respect to the y-axis of the circle $x^2 + y^2 = R^2$ whose mass is m.

10. Find the centre of masses of the figure bounded by the curve $y = 2\sqrt{x}$, x-axis, and the straight line $x = 1$.

11. Find the centre of masses of the figure bounded by the lines $y^2 = 20x$ and $x^2 = 20y$.

12. Find the coordinates of the centre of masses of the parabolic segment bounded by the lines $y = 4 - x^2$ and $y = 0$.

13. Using Guldin's first theorem, find the centre of masses of a semicircle of radius a.

14. Making use of Guldin's theorems, compute the area of the lateral surface and volume of a right cone with altitude H and radius of the base R.

15. Using Guldin's theorems, compute the surface area and volume of the solid generated by rotating an equilateral triangle with side a about an axis located at a distance d ($d > a$) from its centre of masses.

Chapter 9

SERIES

9.1. Number Series

1. A Number Series and its Sum. Suppose we are given a sequence of numbers $\{a_n\}$, $n \in N$, or in the expanded form

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots.$$

The expression

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \text{ or } \sum_{n=1}^{\infty} a_n \tag{1}$$

is called a number series, and the numbers $a_1, a_2, \ldots, a_n, \ldots$ are the terms of series (1), the first term, the second term, and so on, respectively, a_n being called the nth term or the general term of series (1).

The sums $S_1 = a_1$, $S_2 = a_1 + a_2, \ldots, S_n = a_1 + \ldots + a_n, \ldots$ are called partial sums of series (1).

The series

$$\sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \ldots \tag{2}$$

is the kth remainder of series (1). It is otained from series (1) on rejecting first k terms of this series.

A series is said to be convergent if the sequence of its partial sums converges, that is, if there exists a finite limit

$$S = \lim_{k \to \infty} S_k = \lim_{k \to \infty} \sum_{n=1}^{k} a_n.$$

This limit is called the sum of series (1). If (1) is a convergent series and S is its sum, then we shall write

$$S = \sum_{n=1}^{\infty} a_n.$$

A series is said to be divergent if the sequence of its partial sums diverges.

Example 1. Test the series $\sum_{n \to \infty}^{\infty} q^n$ for convergence.

△ By the formula for the sum of a geometric progression with common ratio q, we have

$$S_k = \sum_{n=1}^{k} q^n = \frac{q - q^{k+1}}{1-q}.$$

If $|q| < 1$, then $\lim\limits_{k\to\infty} q^{k+1} = 0$, and therefore

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{q - q^{k+1}}{1-q} = \frac{q}{1-q}.$$

Consequently, if $|q| < 1$, the given series converges to

$$\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}.$$

If $|q| > 1$, then $\lim\limits_{k\to\infty} |q|^{k+1} = +\infty$, therefore the sequence $\{q^{k+1}\}$ is unbounded and, consequently, has no limit. Hence it follows that the sequence $\{S_k\}$ has no limit either, that is, the given series diverges for $|q| > 1$.

Let now $|q| = 1$. If $q = 1$, then

$$S_1 = 1,\ S_2 = 2,\ \ldots,\ S_k = k,\ \ldots,$$

and, consequently, the given series diverges. If $q = -1$, then

$$S_1 = -1,\ S_2 = 0,\ S_3 = -1,\ \ldots,$$

that is, the partial sums with odd subscripts are equal to -1, whereas those with even subscripts are equal to zero. Such a sequence has no limit.

Consequently, if $|q| = 1$, then the series diverges. ▲

Example 2. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ for convergence.

△ This series is called harmonic. Let us show that it diverges. Note that

$$\int_n^{n+1} \frac{dx}{x} < \frac{1}{n} \quad \text{for any } n \in N.$$

Therefore

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \sum_{n=1}^{k} \int_n^{n+1} \frac{dx}{x} = \int_1^{k+1} \frac{dx}{x} \ln x \Big|_1^{k+1} = \ln(k+1).$$

Thus,

$$S_k > \ln(k+1)$$

for any $k \in N$, and therefore $\lim\limits_{k\to\infty} S_k = +\infty$. Consequently, a harmonic series diverges. ▲

2. Basic Theorems on Convergence of Number Series.

Theorem 9.1.1. *If a series converges, then any of its remainders also converges. If some remainder of a series converges, then the series also converges.*

□ Let there be given a series $\sum\limits_{n=1}^{\infty} a_n$. The partial sums S_N and S_{N+k} of this series are related as follows:

$$S_{N+k} = \sum_{n=1}^{N+k} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{N+k} a_n = S_N + S_k^*,$$

where S_k^* is the kth partial sum of the Nth remainder of the series. Let the series be convergent, and let its sum be equal to S. Then

$$\lim_{k\to\infty} S_k^* = \lim_{k\to\infty} (S_{N+k} - S_N) = \lim_{k\to\infty} S_{N+k} - S_N$$
$$= \lim_{m\to\infty} S_m - S_N = S - S_N,$$

and, consequently, the Nth remainder of the series converges. Conversely, if the Nth remainder of the series converges, and its sum is equal to R_N, then

$$\lim_{m\to\infty} S_m = \lim_{k\to\infty} S_{N+k} = \lim_{k\to\infty} (S_N + S_k^*) = S_N + \lim_{k\to\infty} S_k^* = S_N + R_N,$$

and, consequently, the series converges. ■

Theorem 9.1.2. *If two series with the general terms a_n and b_n converge, and*

$$\sum_{n=1}^{\infty} a_n = A, \quad \sum_{n=1}^{\infty} b_n = B.$$

Then for any numbers α and β the series with the general term $c_n = \alpha a_n + \beta b_n$ converges and

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B.$$

□ Indeed,

$$\lim_{k\to\infty} \sum_{n=1}^{k} (\alpha a_n + \beta b_n) = \lim_{k\to\infty} \left(\alpha \cdot \sum_{n=1}^{k} a_n + \beta \sum_{n=1}^{k} b_n \right)$$
$$= \alpha \lim_{k\to\infty} \sum_{n=1}^{k} a_n + \beta \lim_{k\to\infty} \sum_{n=1}^{k} b_n = \alpha A + \beta B. \blacksquare$$

Example 1. Compute the sum of the series $\sum_{n=1}^{\infty} (4\cdot 3^n + 8\cdot 5^{1-n})$.

▲ The general term of this series has the form

$$C_n = 4\left(\frac{1}{3}\right)^n + 8\cdot 5\cdot\left(\frac{1}{5}\right)^n.$$

As it was shown in Example 1 of the preceding subsection, the series with the general terms $a_n = \left(\frac{1}{3}\right)^n$ and $b_n = \left(\frac{1}{5}\right)^n$ converge, and

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2},\quad \sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}.$$

By virtue of Theorem 9.1.2, the given series converges, and

$$\sum_{n=1}^{\infty}(4\cdot 3^{-n} + 8\cdot 5^{1-n}) = 4\cdot\frac{1}{2} + 40\cdot\frac{1}{4} = 12. \ ▲$$

Theorem 9.1.3. (*the necessary condition for a series to converge*). *If a series* $\sum_{n=1}^{\infty} a_n$ *converges, then its general term tends to zero, that is,*

$$\lim_{n\to\infty} a_n = 0.$$

□ Let the series $\sum_{n=1}^{\infty} a_n$ converge and its sum be equal to S. It is easy to see that $a_n = S_n - S_{n-1}$ for any $n \geqslant 2$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0. \ ■$$

Note that the condition $a_n \to 0$ as $n \to \infty$ is necessary but not sufficient for a series to converge. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, considered in Example 2 of the preceding subsection, meets this condition but still diverges.

Example 2. Test the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ for convergence.

△ The given series diverges according to Theorem 9.1.3 since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{2n+1} = \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0. \ ▲$$

3. Series with Nonnegative Terms. The Comparison Test. Note that a sequence of partial sums of any series $\sum\limits_{n=1}^{\infty} a_n$, in which $a_n \geqslant 0$, is nondecreasing. Indeed, since $a_{n+1} \geqslant 0$, we have

$$S_{n+1} = S_n + a_{n+1} \geqslant S_n$$

for any $n \in N$. By Weierstrass's theorem, a nondecreasing bounded sequence has a finite limit. And if it is unbounded, then it diverges (see Chapter 5). Thus, the following theorem holds true.

Theorem 9.1.4. *For a series with nonnegative terms to converge, it is necessary and sufficient that the sequence of its partial sums be bounded.*

Example 1. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ for convergence.

△ Let $\alpha > 1$. Then for any $k \geqslant 2$ we have:

$$S_k = \sum_{n=1}^{k} \frac{1}{n^{\alpha}} = 1 + \sum_{n=2}^{k} \frac{1}{n^{\alpha}} \cdot \int_{n-1}^{n} dx < 1 + \sum_{n=2}^{k} \int_{n-1}^{n} \frac{dx}{x^{\alpha}}$$

$$= 1 + \int_{1}^{k} \frac{dx}{x^{\alpha}} = 1 + \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^k < 1 + \frac{1}{\alpha-1} = \frac{\alpha}{\alpha-1},$$

and, by Theorem 9.1.4, the series converges. For $\alpha = 1$ the series under consideration turns into a harmonic divergent series. For $\alpha < 1$ we have (see Example 2 from subsection 1):

$$S_k = \sum_{n=1}^{k} \frac{1}{n^{\alpha}} > \sum_{n=1}^{k} \frac{1}{n} > \ln(1+k),$$

and therefore $\lim\limits_{k \to \infty} S_k = +\infty$. Consequently, for $\alpha < 1$ the series diverges. ▲

Theorem 9.1.5 (*the comparison test*). *Let for the series* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=}^{\infty} b_n$ *the following condition be satisfied: there is an* N *such t* $\leqslant a_n \leqslant b_n$ *for any* $n \geqslant N$. *Then, if the series* $\sum\limits_{n=1}^{\infty} b_n$

converges, so does the series $\sum_{n=1}^{\infty} a_n$. *And if the series* $\sum_{n=1}^{\infty} a_n$ *diverges, then the series* $\sum_{n=1}^{\infty} b_n$ *also diverges.*

□ If the series $\sum_{n=1}^{\infty} b_n$ converges, so does its remainder $\sum_{n=N}^{\infty} b_n$ (see Theorem 9.1.1 in subsection 2). Let $\sum_{n=N}^{\infty} b_n = B$, then

$$\sum_{n=N}^{k} a_n \leqslant \sum_{n=N}^{k} b_n \leqslant \sum_{n=N}^{\infty} b_n = B$$

for any $k \geqslant N$, that is, the sequence of partial sums of the series $\sum_{n=N}^{\infty} a_n$ is bounded. Since, in addition, $a_n \geqslant 0$ for any $n \geqslant N$, by virtue of Theorem 9.1.4, the series $\sum_{n=N}^{\infty} a_n$ converges. Consequently, the series $\sum_{n=1}^{\infty} a_n$ also converges (see Theorem 9.1.1 in subsection 2). If the series composed of a_n diverges, then the series formed of b_n cannot converge, since from the assumption of its convergence there immediately follows the convergence of the series made up of a_n which will contradict the condition of its divergence. ■

Example 2. Test the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^4}$ for convergence.

△ Since $0 \leqslant \frac{\cos^2 n}{n^4} \leqslant \frac{1}{n^4}$ for any $n \in N$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges (since $\alpha = 4 > 1$, see Example 1), by the comparison test, the given series converges. ▲

Example 3. Test the series $\sum_{n=1}^{\infty} \frac{\log n}{n}$ for convergence.

△ Since $\frac{\log n}{n} > \frac{1}{n}$ for any $n \geqslant 11$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test, the given series also diverges. ▲

4. D'Alembert's and Cauchy's Tests.

Lemma 1. *Let there be given a series* $\sum_{n=1}^{\infty} a_n$ *with positive terms. Then if there is a number* q *such that*

$$\frac{a_{n+1}}{a_n} \leqslant q < 1 \tag{1}$$

for any $n \in N$, *then series converges. And if*

$$\frac{a_{n+1}}{a_n} \geqslant 1 \tag{2}$$

for any $n \in N$, *then the series diverges.*

□ Let condition (1) be satisfied. Then

$$a_{n+1} \leqslant qa_n \leqslant q^2 a_{n-1} \leqslant \ldots \leqslant a_1 q^n,$$

whence

$$a_n \leqslant a_1 q^{n-1}$$

for any $n \in N$. Since the series $\sum_{n=1}^{\infty} a_1 q^{n-1} = \frac{a_1}{q} \cdot \sum_{n=1}^{\infty} q^n$ converges in the case $0 < q < 1$, by the comparison test, the given series $\sum_{n=1}^{\infty} a_n$ also converges.

Let condition (2) be satisfied. Then

$$a_{n+1} \geqslant a_n \geqslant \ldots \geqslant a_1,$$

that is, $a_n \geqslant a_1 > 0$ for any $n \in N$. Hence it follows that the general term of the series does not tend to zero, that is, the necessary condition for convergence is not satisfied for the series, and, consequently the series diverges. ■

Theorem 9.1.6 (*D'Alembert's test*). *Let for a series* $\sum_{n=1}^{\infty} a_n$ *with positive terms* a_n *the following condition be satisfied:*

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = q.$$

Then the series is convergent for $q < 1$ *and divergent for* $q > 1$.

□ According to the definition of the limit of a sequence, we have that for any $\varepsilon > 0$ there is a number N such that

$$q - \varepsilon < \frac{a_{n+1}}{a_n} < q + \varepsilon$$

for any $n > N$. If $q < 1$, then, choosing $\varepsilon > 0$ so that $q + \varepsilon < 1$, we get

$$\frac{a_{n+1}}{a_n} < q + \varepsilon < 1$$

for any $n \geqslant N$, that is, the Nth remainder of the given series converges since condition (1) of Lemma 1 is satisfied for its terms. Consequently, according to Theorem 9.1.1, the given series converges.

If $q > 1$, then, choosing $\varepsilon > 0$ so that $q - \varepsilon > 1$, we get

$$\frac{a_{n+1}}{a_n} > q - \varepsilon > 1$$

for any $n \geqslant N$, that is, the Nth remainder meets condition (2) of the lemma and therefore diverges. Consequently, the given series also diverges by virtue of Theorem 9.1.1. ■

Note. If the limit $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist or is equal to unity, then D'Alembert's test gives no answer to the question on the convergence of a series.

Example 1. Test the series $\sum\limits_{n=1}^{\infty} \frac{n}{4^n}$ for convergence.

△ Here $a_n = \frac{n}{4^n}$ and $a_{n+1} = \frac{n+1}{4^{n+1}}$. Therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^n (n+1)}{4^{n+1} \cdot n} = \frac{1}{4} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{4} < 1,$$

and, according to D'Alembert's test, the given series is convergent. ▲

Example 2. Test the series $\sum\limits_{n=1}^{\infty} \frac{2^n}{n}$ for convergence.

△ Here $a_n = \frac{2^n}{n}$ and $a_{n+1} = \frac{2^{n+1}}{n+1}$. Therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n}{n+1} = 2 \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 2 > 1,$$

and, according to D'Alembert's test, the series is divergent. ▲

Example 3. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ for convergence.

△ Here $a_n = \frac{1}{n(n+1)}$ and $a_{n+1} = \frac{1}{(n+1)(n+2)}$. Therefore

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1,$$

and, on the basis of D'Alembert's test, we cannot conclude whether the given series is convergent or divergent. Let us prove that it

is convergent using the definition of the convergence of a series. Since

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1},$$

we have

$$S_k=\sum_{n=1}^{k} a_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$=1-\frac{1}{k+1},$$

whence

$$\lim_{k\to\infty} S_k=\lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)=1.$$

Thus, the given series converges, and its sum is equal to 1. ■

Lemma 2. *Let a series* $\sum_{n=1}^{\infty} a_n$ *be given with nonnegative terms. Then, if there is a* q *such that* $\sqrt[n]{a_n}\leqslant q<1$ *for any* $n\geqslant N$, *then the series converges. And if* $\sqrt[n]{a_n}\geqslant 1$, *then the series diverges.*

□Since, by hypothesis, $\sqrt[n]{a_n}\leqslant q<1$ for any $n\geqslant N$, we have $a_n\leqslant q^n$ $(q<1)$ for any $n\geqslant N$. By virtue of the comparison test, the last inequality (see Theorem 9.1.5) implies that the given series converges since the series $\sum_{n=1}^{\infty} q^n$ $(q<1)$ converges.

Let now $\sqrt[n]{a_n}\geqslant 1$ for any $n\geqslant N$. Then $a_n\geqslant 1^n$ for any $n\geqslant N$. But the series $\sum_{n=1}^{\infty} 1^n$ diverges (its general term does not tend to zero as $n\to\infty$).
Therefore, by the comparison test, the given series also diverges. ■

Theorem 9.1.7. (*Cauchy's test*). *Let for a series* $\sum_{n=1}^{\infty} a_n$ *with nonnegative terms the following condition be satisfied:*

$$\lim_{n\to\infty}\sqrt[n]{a_n}=q.$$

Then if $q<1$, *the series converges, and if* $q>1$, *the series diverges.*

□By the definition of limit, we have that for any $\varepsilon>0$ there is an N such that

$$q-\varepsilon<\sqrt[n]{a_n}<q+\varepsilon \qquad (1)$$

for any $n \geqslant N$. If $q < 1$, then, choosing $\varepsilon > 0$ so that $q + \varepsilon < 1$, from inequality (1) we get that $\sqrt[n]{a_n} < q + \varepsilon < 1$ for any $n \geqslant N$. Consequently, by Lemma 2, the given series will converge.

Let now $q > 1$, then, choosing $\varepsilon > 0$ so that $q - \varepsilon > 1$, from inequality (1) we have that for any $n \geqslant N$

$$\sqrt[n]{a_n} > q - \varepsilon > 1.$$

According to Lemma 2, this means that the given series diverges. ■

Example 4. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^n}$ for convergence.

△ Here $a_n = \frac{1}{n^n}$. Therefore

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1,$$

and, by Cauchy's test, the series converges. ▲

Example 5. Test the series $\sum\limits_{n=1}^{\infty} \left(\frac{3n+1}{n}\right)^n$ for convergence.

△ Here $a_n = \left(\frac{3n+1}{n}\right)^n$. Therefore

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{3n+1}{n} = 3 > 1,$$

and, by Cauchy's test, the given series diverges. ▲

Note. If $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ does not exist or is equal to unity, then Cauchy's test provides no answer to the question concerning the convergence of a series.

5. Absolutely and Conditionally Convergent Series. Leibniz' Test. So-called absolutely convergent series whose terms are any real numbers constitute an important class of convergent series.

Definition 1. The series $\sum\limits_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges.

Note that here nothing is said about the convergence of the original series. Such an assumption would be needless, since the following theorem holds true.

Theorem 9.1.8. *If the series* $\sum\limits_{n=1}^{\infty} |a_n|$ *converges, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *also converges.*

□ We set

$$p_n = \frac{|a_n| + a_n}{2} \text{ and } q_n = \frac{|a_n| - a_n}{2}.$$

The following inequalities are obviously valid:

$$0 \leqslant p_n \leqslant |a_n|, \quad 0 \leqslant q_n \leqslant |a_n|$$

for any $n \in N$. It follows from the last inequalities that the series $\sum\limits_{n=1}^{\infty} p_n$ and $\sum\limits_{n=1}^{\infty} q_n$ converge by virtue of the comparison test. Since $a_n = p_n - q_n$, the given series also converges (see Theorem 9.1.2). ■

This theorem states that if a series absolutely converges, then it simply converges.

To test series for absolute convergence, we can use all the convergence tests for series with nonnegative terms.

Example 1. Test the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ for absolute convergence.

△ Since

$$a_n = \frac{(-1)^{n-1}}{n^3}, \quad |a_n| = \frac{1}{n^3},$$

and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ converges, the given series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely. ▲

Example 2. Test the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ for absolute convergence.

△ Since

$$a_n = \frac{(-1)^n}{\sqrt{n}}, \quad |a_n| = \frac{1}{n^{\frac{1}{2}}},$$

and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges, the given series is not absolutely convergent. ▲

Consider alternating series whose terms are alternately positive and negative.

Theorem 9.1.9 (*Leibniz' test*). *If a sequence* $\{a_n\}$ *of positive numbers monotonically decreases, and* $\lim\limits_{n\to\infty} a_n = 0$, *then the alternating series* $\sum\limits_{n=1}^{\infty} (-1)^{n-1}\cdot a_n$ *converges.*

□ For partial sums with even subscripts we have:

$$S_{2k} = a_1 - a_2 + \ldots + a_{2k-1} - a_{2k} = \sum_{n=1}^{k} (a_{2n-1} - a_{2n}).$$

Hence, according to the inequality $a_{2n-1} - a_{2n} \geqslant 0$, it follows that S_{2k}, $k \in N$, is a monotonically increasing sequence. On the other hand,

$$S_{2k} = a_1 - (a_2 - a_3) - \ldots - a_{2k} < a_1$$

for any $k \in N$, that is the sequence $\{S_{2k}\}$ is bounded. Consequently, it has a limit. Let

$$\lim_{k\to\infty} S_{2k} = S. \tag{1}$$

For partial sums with odd subscripts we have:

$$\lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} (S_{2k} - a_{2k}) = \lim_{k\to\infty} S_{2k} - \lim_{k\to\infty} a_{2k} = S. \tag{2}$$

From relations (1) and (2) it follows that the given series converges, and its sum is equal to S. ■

Example 3. Test the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}$ for convergence.

△ Here $a_n = \frac{1}{n^\alpha}$. Since $\{a_n\}$ is a monotonically decreasing sequence, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0$, by Leibniz' test, the given series converges for any $\alpha > 0$. Since $\left|\frac{(-1)^n}{n^\alpha}\right| = \frac{1}{n^\alpha}$ and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for $\alpha > 1$ and diverges for $0 < \alpha \leqslant 1$ (see Example 1 in subsection 3), the given series is absolutely convergent for $\alpha > 1$ and is not absolutely convergent for $0 < \alpha \leqslant 1$. ▲

Definition 2. A series is said to be conditionally convergent if it is convergent but not absolutely.

Example 4. Test the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^8}$ and $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ for convergence.

△ Both series converge by Leibniz' test. In this case, the first series is absolutely convergent, whereas the second series is not absolutely convergent. Consequently, it is conditionally convergent. ▲

Absolutely convergent series possess many properties inherent in ordinary finite sums. In particular, among them is a property analogous to commutativity: any rearrangement of the terms of an absolutely convergent series does not affect its sum.

Conditionally convergent series essentially differ from ordinary finite sums by their properties. For instance, the following statement holds true for them: the terms of a conditionally convergent series can be rearranged so that the series thus obtained will converge to any preassigned number. Moreover, they can be rearranged so that the newly obtained series will diverge.

Exercises

1. Evaluate the sums of the given series:

(a) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)$; (b) $\sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$;

(c) $\sum\limits_{n=1}^{\infty} \frac{5}{10^n}$; (d) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{7^n}$; (e) $\sum\limits_{n=1}^{\infty} \left(\frac{2}{10^{2n}} + \frac{50}{10^{2n+1}}\right)$.

2. Test the following series for convergence:

(a) $\sum\limits_{n=1}^{\infty} \frac{n}{100n+1}$; (b) $\sum\limits_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$; (c) $\sum\limits_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$.

3. Using the comparison test, investigate the given series for convergence:

(a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+3}$; (b) $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt[2]{n}-1}$; (c) $\sum\limits_{n=2}^{\infty} \frac{1}{\ln n}$;

(d) $\sum\limits_{n=1}^{\infty} 2^n \sin\frac{1}{3^n}$; (e) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+2)}}$; (f) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$.

4. Using D'Alembert's test, investigate the following series for convergence:

(a) $\sum\limits_{n=1}^{\infty} \frac{n}{(2n-1)!}$; (b) $\sum\limits_{n=1}^{\infty} \frac{3^n}{n!}$; (c) $\sum\limits_{n=1}^{\infty} \frac{n}{3^n}$; (d) $\sum\limits_{n=1}^{\infty} \frac{2n-1}{2^n}$;

(e) $\sum\limits_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$.

5. Test the following series for absolute and conditional convergence:

(a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$; (b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n-1}$; (c) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n!}$;

(d) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$; (e) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$; (f) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n+1}}$.

9.2. Power Series

1. The Notion of Power Series. The Radius of Convergence. Abel's Theorem. Let a sequence of functions $a_1(x), a_2(x), \ldots, a_n(x), \ldots$, be given which is defined on the same set A. Series of the form

$$\sum_{n=1}^{\infty} a_n(x) = a_1(x) + a_2(x) + \ldots + a_n(x) + \ldots$$

are called functional, since its terms are functions of x. With x fixed, we can obtain various series from the set A, which may turn out both convergent and divergent. The totality of those values $x \in A$ for which a functional series converges is called the domain of convergence of a series. In the domain of convergence of a functional series its sum is a certain function of x: $S = S(x)$.

Example 1. Find the domain of convergence of the series $\sum\limits_{n=0}^{\infty} x^n$.

△ This series converges for all $x \in (-1, 1)$. Indeed, for each $x \in (-1, 1)$ the sum of the series is equal to $\frac{1}{1-x}$ (the sum of a geometric progression). Thus, in the interval $(-1, 1)$ the given series defines the function

$$S(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. \ ▲$$

Consider functional series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \tag{1}$$

where x_0 and $a_0, a_1, \ldots, a_n, \ldots$ are given real numbers, and $x \in \boldsymbol{R}$ is a real variable. Such series are called power series. The numbers $a_0, a_1, \ldots, a_n, \ldots$ are called the coefficients of the power series (1). For each value of the variable x series (1) is a number series.

Note that in a power series the terms are counted beginning not from unity but from zero: the first term is called the zero term, the second term—the first term, and so on. For a power series such counting seems to be natural, since the zeroth term is the product of the coefficient a_0 by the polynomial of zero degree $(x - x_0)^0 = 1$, the first term is the product of a_1 by $(x - x_0)$, and, in general, the nth term is equal to the product of the coefficient a_n by the polynomial of degree n $(x - x_0)^n$.

If series (1) converges for $x = x_1$, then this power series is said to converge at the point x_1. The set of all points x at which series (1) converges is called the domain of convergence of the power series. Note that any power series of the form (1) converges at the point $x = x_0$. Therefore the domain of convergence of a power series always contains at least one point. Further information concerning the domain of convergence of a power series can be obtained from the following theorem named after the Norwegian mathematician Niels Henrik Abel.

Theorem 9.2.1. *If a power series* (1) *converges at a point* $x_1 \neq x_0$ *then it converges absolutely at any point* x *such that* $|x - x_0| < |x_1 - x_0|$. *But if series* (1) *diverges at the point* x_1, *then it diverges at any point* x *for which* $|x - x_0| > |x_1 - x_0|$.

□ Let the number series $\sum_{n=0}^{\infty} a_n (x_1 - x_0)^n$ converge. Then its general term is bounded since $\lim_{n \to \infty} a_n (x_1 - x_0)^n = 0$. Let $|a_n (x_1 - x_0)^n| \leqslant M$ for all n. Consequently, if $|x - x_0| < |x_1 - x_0|$, then

$$|a_n (x - x_0)^n| = |a_n (x_1 - x_0)^n| \cdot \left(\frac{|x - x_0|}{|x_1 - x_0|} \right)^n \leqslant Mq^n,$$

where $q = \frac{|x - x_0|}{|x_1 - x_0|} < 1$. Applying the comparison test (with a geometric progression, we see that at any point x satisfying the inequality $|x - x_0| < |x_1 - x_0|$ the series converges.

Let now the series $\sum_{n=0}^{\infty} a_n (x_1 - x_0)^n$ diverge. Then series (1) diverges at any point x for which $|x - x_0| > |x_1 - x_0|$. Indeed, if

we assume that it converges at the point x, then, according to our proof, it will also converge at the point x_1. But this contradicts the fact that at the point x_1 the series diverges. ■

Hence, from Abel's theorem it follows that for a power series (1) three cases are possible:

(1) series (1) converges only at the point x_0;

(2) series (1) converges at all $x \in R$;

(3) there exists a number $R > 0$ such that for all x from interval $(x_0 - R, x_0 + R)$ series (1) converges, and for all x, $| x - x_0 | > R$, the series diverges.

Definition 1. The number $R > 0$ such that series (1) converges for all x, $| x - x_0 | < R$ and diverges for all x, $| x - x_0 | > R$ is called the radius of convergence of series (1).

If series (1) converges only at the point x_0, then, by definition, $R = 0$. If series (1) converges at any $x \in R$, then, by definition, $R = +\infty$.

Theorem 9.2.2. *If in power series* (1) *the sequence* $\left\{\frac{|a_n|}{|a_{n+1}|}\right\}$ *has either a finite or an infinite limit, then the following formula is valid for the radius of convergence*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}. \tag{2}$$

□ Let

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = A.$$

First we consider the case when $0 < A < +\infty$. We choose any $x_1 \neq x_0$ and apply D'Alembert's test to the number series $\sum_{n=0}^{\infty} | a_n | | x_1 - x_0 |^n$. Here the general term of the series $u_n = | a_n | | x_1 - x_0 |^n$, and therefore

$$\lim_{n\to\infty} \frac{u_{n+1}}{n_n} = \lim_{n\to\infty} \frac{|a_{n+1}| |x_1 - x_0|^{n+1}}{|a_n| |x_1 - x_0|^n}$$

$$= |x_1 - x_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x_1 - x_0| \frac{1}{A}.$$

According to D'Alembert's test, if $\frac{1}{A} |x_1 - x_0| < 1$, that is, $|x_1 - x_0| < A$, then series (1) converges absolutely, and if $|x_1 - x_0| > A$, the series diverges. Consequently, $A = R$.

If $A = +\infty$, then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$ for any $x_1 \neq x_0$, and, consequently, the series converges for any x, that is, $R = +\infty$. If

$A=0$, then $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=+\infty>1$ for any $x_1\neq x_0$, and, consequently, the series diverges for any $x\neq x_0$, and this means that $R=0$. ■

Applying Cauchy's test for investigating number series for convergence, we can establish the validity of the following theorem:

Theorem 9.2.3. *Let the sequence* $\{\sqrt[n]{|a_n|}\}$ *have a finite or infinite limit. Then for the radius of convergence the following formula is valid:*

$$R=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}. \tag{3}$$

Example 2. Find the radii of convergence of the series $\sum\limits_{n=0}^{\infty}x^n$ and $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$.

△ It was shown in the preceding subsection that the first series converges on the interval $(-1, 1)$, that is, for all x, $|x|<1$, and, consequently, we may assume that $R=1$. Indeed, making use of formula (2), we get

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{1}{1}=1.$$

By formula (2), we get for the second series:

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)=+\infty. \blacktriangle$$

Example 3. Find the domain of convergence of the series $\sum\limits_{n=0}^{\infty}\frac{x^n}{n^2+1}$.

△ Since $a_n=\frac{1}{1+n^2}$, by formula (2), we get:

$$R=\lim_{n\to\infty}\frac{(n+1)^2+1}{n^2+1}=\lim_{n\to\infty}\frac{n^2+2n+2}{n^2+1}=1.$$

Thus, the series converges on the interval $(-1, 1)$. Since the number series $\sum\limits_{n=0}^{\infty}\frac{1}{n^2+1}$ converges, the given series converges absolutely for $x=\pm1$. Finally, the given series converges absolutely on the interval $[-1, 1]$ and diverges for all $|x|>1$. ▲

Example 4. Find the domain of convergence of the series $\sum\limits_{n=0}^{\infty}\frac{x^n}{n+2}$.

△ Since $a_n = \frac{1}{n+2}$, by formula (2), we get:

$$R = \lim_{n\to\infty} \frac{n+3}{n+2} = 1.$$

At the point $x = 1$ the series diverges since it turns into the divergent numbers series $\sum_{n=0}^{\infty} \frac{1}{n+2}$. At the point $x = -1$ the series converges by Leibniz' test, since it has the form $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$. Finally, the given series converges on the interval $[-1, 1)$, converges absolutely on the interval $(-1, 1)$, and diverges for all x, $|x| > 1$. ▲

Example 5. Find the domain of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$.

△ Using formula (3), we find the radius of convergence. Since $a_n = \frac{1}{3^n}$, we have

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{3^n}}} = 3.$$

Thus, the series converges on the interval $(-3, 3)$. Since at the points $x = -3$ and $x = 3$ the given series turns into divergent series of the form $\sum_{n=0}^{\infty} (-1)^n$ and $\sum_{n=0}^{\infty} 1^n$ (the general terms of these series do not tend to zero), the given series converges absolutely on the interval $(-3, 3)$ and diverges for all $x \geqslant 3$. ▲

2. Differentiation and Integration of Power Series. On the convergence interval the power series defines the function

$$S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{1}$$

This function is called the sum of the power series. It is possible to prove that the sum of the power series is continuous and has continuous derivatives of any order on the convergence interval of the series. In addition, the derivative of the sum $S(x)$ is the sum of the series made up of derivatives, that is,

$$S'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}. \tag{2}$$

Similarly, we can easily prove the following:

$$S''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-x_0)^{n-2},$$

$$S'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)\, a_n (x-x_0)^{n-3}$$

and so forth. In this sense, we sometimes say that for power series the derivative of the sum is equal to the sum of the derivatives or that it is possible to differentiate power series termwise:

$$\left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)' = \sum_{n=0}^{\infty} (a_n (x-x_0)^n)'. \tag{3}$$

We also take without proof the assertion on termwise integration of power series: the integral of the sum of a power series is equal to the sum of the series made up of the integrals of the corresponding terms of the given series, that is,

$$\int_a^b S(x)\, dx = \sum_{n=0}^{\infty} \int_a^b a_n (x-x_0)^n\, dx \tag{4}$$

for any interval $[a, b]$ from the convergence interval. In other words, on the convergence interval a power series can be integrated termwise:

$$\int_a^b \sum_{n=0}^{\infty} a_n (x-x_0)^n\, dx = \sum_{n=0}^{\infty} \int_a^b a_n (x-x_0)^n\, dx. \tag{5}$$

Example 1. Find the sum of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

△ We find the radius of convergence of the given series. Since $a_n = \frac{(-1)^{n-1}}{n}$, we have $R = \lim\limits_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim\limits_{n\to\infty} \frac{n+1}{n} = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$.

Consequently, the given series converges on the interval $(-1, 1)$, and its sum $S(x)$ has a continuous derivative $S'(x)$. According to formula (2) for termwise differentiation of a power series,

$$S'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{x^n}{n}\right)' = \sum_{n=1}^{\infty} (-x)^{n-1}.$$

The last series on the interval $(-1, 1)$ converges to the function $\frac{1}{1+x}$ (the sum of a geometric progression), and therefore

$$S'(x) = \frac{1}{1+x}. \tag{6}$$

Integrating equality (6), we get:

$$S(x) = \ln(1+x) + C,$$

where C is some constant. Since $S(0) = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{0^n}{n}\right) = 0$ and $\ln(1+0) = 0$, then also $C = 0$. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x)$, $x \in (-1, 1)$. ▲

Example 2. Find the sum of the power series $\sum_{n=1}^{\infty} nx^n$.

△ Here $a_n = n$, therefore, by formula (2) in the preceding subsection we have

$$R = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1,$$

that is, the given series converges on the interval $(-1, 1)$. Further,

$$\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \ldots = x(1 + 2x + 3x^2 + \ldots)$$

$$= x \sum_{n=1}^{\infty} nx^{n-1} = x \sum_{n=1}^{\infty} (x^n)'.$$

By formula (3) we have:

$$\sum_{n=1}^{\infty} (x^n)' = \left(\sum_{n=1}^{\infty} x^n\right)',$$

whence, according to the equality $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ (the sum of a geometric progression), we finally find:

$$\sum_{n=1}^{\infty} nx^n = x\left(\sum_{n=1}^{\infty} x^n\right)' = x\left(\frac{x}{1-x}\right)' = \frac{x}{(1-x)^2}. \text{ ▲}$$

3. Taylor's Series of Elementary Functions. Let a function $f(x)$ be defined and have derivatives of all orders in some neighbourhood of a point x_0. Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tag{1}$$

is called the Taylor's series of the function $f(x)$ at the point x_0.

Lemma. In order for a Taylor's series of a function $f(x)$ to converge to $f(x)$ at a point x, it is necessary and sufficient that at this point the remainder of Taylor's formula tend to zero as $n \to \infty$.

□ From the Taylor formula (see 6.11)

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + r_n(x)$$

it follows that if $r_n(x) \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = \lim_{n\to\infty} (f(x) - r_n(x)) = f(x),$$

that is, Taylor's series (1) converges to $f(x)$. Conversely, if series (1) converges to $f(x)$, then

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty} \left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \right)$$

$$= f(x) - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = 0. \blacksquare$$

Theorem 1. *The Taylor's series of the function e^x converges to e^x for any $x \in \boldsymbol{R}$, that is,*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \boldsymbol{R}. \tag{2}$$

□ Since for the given power series $a_n = \frac{1}{n!}$, by formula (2) in subsection 1, we find the radius of convergence of series (2):

$$R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = +\infty.$$

Thus, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges everywhere on the number line, that is, for any $x \in \boldsymbol{R}$. Making use of the above lemma, we further prove that series (2) has the function e^x, $x \in \boldsymbol{R}$, as its sum. Taylor's formula with a remainder in Lagrange's form for the function e^x has the form (see 6.11):

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!} e^c,$$

where c lies between x and zero. According to the lemma, in order to prove the convergence of the Taylor's series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ to the function e^x for any $x \in \boldsymbol{R}$, we have to prove that

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} e^c = 0 \tag{3}$$

for any $x \in \boldsymbol{R}$. For any fixed $x \in \boldsymbol{R}$ there is an integral $N > 0$, such that $|x| < N$. We denote: $q = \frac{|x|}{N} < 1$. Then for any $n > N$ we shall have:

$$\begin{aligned}\left|\frac{x^{n+1}}{(n+1)!}\right| &= \left|\frac{x}{1}\cdot\frac{x}{2}\cdots\frac{x}{n}\cdot\frac{x}{n+1}\right| \\ &= \left|\frac{x}{1}\right|\left|\frac{x}{2}\right|\cdots\left|\frac{x}{N-1}\right|\left|\frac{x}{N}\right|\cdots\left|\frac{x}{n+1}\right| \\ &< \frac{|x|}{1}\frac{|x|}{2}\cdots\frac{|x|}{N-1}qq\ldots q = \frac{|x|^{N-1}}{(N-1)!}q^{n-N+2}. \end{aligned} \tag{4}$$

Since c lies between x and zero, with x fixed, the quantity e^c is bounded, and the quantity $\frac{|x|^{N-1}}{(N-1)!}$ is constant, that is, it is independent of n, and $q^{n-N+2} \to 0$ as $n \to \infty$, since $0 \leqslant q < 1$. Thus, inequality (4) implies the validity of formula (3), whence, according to the lemma, follows the validity of formula (2). ■

The following two theorems are proved in a similar way.

Theorem 9.2.4. *The Taylor's series of the function sin x converges to sin x for any $x \in \boldsymbol{R}$, that is,*

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \boldsymbol{R}. \tag{5}$$

Theorem 9.2.5. *The Taylor's series of the function* cos x *converges to* cos x *for any $x \in \boldsymbol{R}$, that is,*

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \boldsymbol{R}. \tag{6}$$

As distinct from the considered functions e^x, sin x, and cos x for which Taylor's series converged at the point $x_0 = 0$ for any $x \in \boldsymbol{R}$, the corresponding Taylor's series for the function ln $(1 + x)$ converges only on the interval $(-1, 1)$.

Theorem 9.2.5. *The Taylor's series of the function* $\ln(1+x)$ *converges to* $\ln(1+x)$ *for any* $x \in (-1, 1)$, *that is,*

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad x \in (-1; 1). \tag{7}$$

□ In Example 1 of the preceding subsection it was proved that power series (7) converges for any x from the interval $(-1, 1)$, and its sum is equal to $\ln(1+x)$. ■

Example 1. Using formulas (5) and 6), show that $(\sin x)' = \cos x$.

△ Differentiating termwise power series (5), we find

$$(\sin x)' = \Big(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\Big)'$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^{2n+1})'$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x, \; x \in \boldsymbol{R}. \; ▲$$

Example 2. Expand the function $\sin(x^2)$, $x \in \boldsymbol{R}$, into a Taylor's series.

△ Substituting x^2 for x into the Taylor's formula for the function $\sin x$ (see 6.11):

$$\sin x = \sum_{k=0}^{n} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!} + \frac{(-1)^{n+1} x^{2n+3} \cos c}{(2n+3)!},$$

we find:

$$\sin(x^2) = \sum_{k=0}^{n} (-1)^k \cdot \frac{x^{4k+2}}{(2k+1)!} + \frac{(-1)^{n+1} x^{4n+6} \cos c}{(2n+3)!}. \tag{8}$$

From the fact that the inequality

$$\left| \frac{(-1)^{n+1} x^{4n+6} \cos c}{(3n+3)!} \right| \leqslant \frac{(x^2)^{2n+3}}{(2n+3)!}$$

is valid, it follows that the remainder in Taylor s formula (8) tends to zero as $n \to \infty$. Thus, the Taylor's series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \tag{9}$$

converges to the function $\sin(x^2)$ for any $x \in \boldsymbol{R}$. The radius of convergence of series (9) is equal to infinity:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(2n+3)!}{(2n+1)!} = \lim_{n\to\infty} (2n+2)(2n+3) = +\infty. \blacktriangle$$

4. Evaluating "Inexpressible" Integrals with the Help of Power Series. One of the antiderivatives $F_0(x)$ of the elementary function $\sin(x^2)$ can be written with the help of the definite integral with a variable upper limit (see 8.2):

$$F_0(x) = \int_0^x \sin(u^2)\,du, \tag{1}$$

which belongs to the class of "inexpressible" integrals (see 7.3). The thing is, that the antiderivatives of the function $\sin(x^2)$ are not elementary functions, and we fail to represent integral (1) in the form of a finite combination of elementary functions. However, it can be evaluated with the help of infinite power series. In Example 2 from the preceding subsection it was found that

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}, \quad x \in \boldsymbol{R}.$$

Since a power series can be integrated termwise, we have

$$\int_0^x \sin(u^2)\,du = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{u^{4n+2}}{(2n+1)!}\,du = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x u^{4n+2}\,du$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{u^{4n+3}}{(4n+3)}\Bigg|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2+1)!\,(4n+3)}.$$

Thus, for the antiderivative $F_0(x)$ we get an expression in the form of a power series:

$$F_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)}, \quad x \in \boldsymbol{R}. \tag{2}$$

From formula (2) there follows the following expression for the family of antiderivatives of the function $\sin(x^2)$, that is, for the indefinite integral $\int \sin(x^2)\,dx$:

$$\int \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2+1)!\,(4n+3)} + C, \quad x \in \boldsymbol{R}, \tag{3}$$

where C is a constant.

Example 1. Evaluate the definite integral $\int_0^1 \sin(x^2)\,dx$ with the help of power series.

△ From formula (3) we find that

$$\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+3)} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \ldots . \tag{4}$$

Formula (4) specifies a rapidly convergent series. Indeed, if we confine ourselves to the first four terms of the series written out in formula (4) and evaluate the integral on a microcalculator, then we get

$$\int_0^1 \sin(x^2)\,dx \approx 0.31026816,$$

where the first six digits are significant (not counting the zero)! For comparison, we note that in evaluating the given integral by the rectangle or trapezoid method (8.4), such an accuracy is attained only with a number of terms $N > 200$. ▲

Example 2. Write the program for evaluating the definite integral $\int_0^A \sin(x^2)\,dx$, $A > 0$ on a computer.

△ To evaluate the integral, let us make use of formulas (1) and (2):

$$\int_0^A \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n A^{4n+3}}{(2n+1)!\,(4n+3)} = \frac{A^3}{3} - \frac{A^7}{42} + \frac{A^{11}}{1320} - \frac{A^{15}}{75600} + \ldots . \tag{5}$$

If $0 < A \leqslant 1$, then formula (5) specifies a rapidly converging series, and the given integral can be evaluated on a microcalculator with a high accuracy. And if $A > 1$, the rapidity of convergence of series (5) is considerably slowed down due to a rapid increase in the numerators of the fractions. In this case, to attain the preassigned accuracy of computations, we have to sum up a great number of terms which cannot be done without a computer.

Let us make more precise the algorithm of computations. For alternating series the following assertion holds: the error of computing

a series when retaining a finite number of terms does not exceed in absolute value the magnitude of the first rejected term of the series (if the terms of the series decrease in absolute value). Therefore, when computing the next term of the series, we shall compare its absolute value with the preassigned accuracy of computations. If it turns out that this quantity is greater than the accuracy given, then we shall add the computed term to the current value of the sum and pass over to computing the next term of the series. If otherwise, we shall cease the computing process, since the required accuracy is already achieved, and it only remains to display the results of computations.

To simplify the computation of the expression

$$W = \frac{(-1)^n A^{4n+3}}{(2n+1)!}$$

it is convenient to use the value of this expression computed before and corresponding to the preceding value of n. As a result, the program in BASIC has the form:

```
 1 INPUT E, A
 2 LET W = A*A*A
 3 LET R = W * A
 4 LET S = W/3
 5 LET N = 0
 6 LET N = N + 1
 7 LET W = -W * R/(2 * N * (2 * N + 1))
 8 LET T = W/(4 * N + 3)
 9 IF ABS (T) < E THEN 12
10 LET S = S + T
11 GOTO 6
12 PRINT "N = "; N; "S = "; S
13 END
```

Let us explain this program. The first line puts in the accuracy of computations E and the limit of integration A. Lines 2-5 appropriate the initial values to the variables and determine the constant R = A^4. Increasing the dummy index N by unity in the sixth line, we compute the new value of the expression W in the seventh line and a new term of the series T in the eighth line. The ninth line compares the quantities | T | and E. If | T | < E, we stop computing and type the result of computations, that is, the sum S (writing the explanatory inscription "S = " before it) and the number of summed up terms of the series equal to N. If | T | > E, that is, the required accuracy is not achieved, then we add the computed term of the series T to the current value of the sum S and repeat the calcula-

tions. We put the ready program in the computer storage and, with the sign? appearing on the display of the computer, set in the value of E equal to 0.001 and the value of A equal to 2. In some time the result of computations appears on the display:

$$N = 6 \quad S = 0.8040285,$$

where, according to the preassigned accuracy, three digits after the decimal point are correct. If now we repeat the computations retaining the value of E and put in A = 3, then we get:

$$N = 13 \quad S = 0.7738290.$$

Note that this time we have had to sum up a greater number of terms of the series. In any case, the volume of computations by formula (5) required for attaining the preassigned accuracy is considerably less than in the case of computing a given integral by the rectangle or trapezoid formula (see 8.4). However, the computations by formula (5) for sufficiently large A ($A \approx 5$) yield first inexact and then (for still greater values of A, when $A > 5$) simply incorrect results. In particular, assigning E = 0.001 and A = 10, we get a "fantastic" result:

$$N = 136 \quad S = -0.621 \cdot 10^{30}, \tag{6}$$

whereas the computations by the trapezoid formula (these computations are always possible) yield a correct result:

$$\int_0^{10} \sin(x^2)\,dx \approx 0.58.$$

This is explained by the fact that series (5) which converges for any $A \in R$ is practically impossible to be computed for large values of | A | since computations are carried out with a finite number of digits after the decimal point. For large values of | A | the terms of alternating series (5) first abruptly increase and then begin to decrease tending to zero in the limit. On most computers using programs written in BASIC computations are performed with seven digits after the decimal point. As a result, a subsequent computation of the terms of series (5) and their summation lead to obtaining an absolutely wrong result (6). In more detail, this topic is discussed in special courses of numerical methods of computation. ▲

5. **Power Series with Complex Terms. Euler's Formula.** Up till now we have considered power series with real terms. However, it is also possible to consider power series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{1}$$

where z_0, a_0, a_1, . . ., a_n, . . . are given complex numbers, and the variable z takes on complex values. Series (1) is said to converge at the point z_1 if there is a complex number S, called the sum of the series, such that for any real $\varepsilon > 0$ there is a number N such that for any $k > N$ the inequality

$$\left| \sum_{n=0}^{k} a_n (z_1 - z_0)^n - S \right| < \varepsilon$$

will be satisfied. The set of all z for which series (1) converges is called the domain of convergence of series (1). Note that power series (1) always converges at the point $z = z_0$, so that the domain of convergence contains at least one point.

As above, we can prove that if series (1) converges at the point $z_1 \neq z_0$, then it converges at any point z satisfying the inequality $| z - z_0 | < | z_1 - z_0 |$. And if it diverges at the point z_1, then it also diverges at any point z satisfying the inequality $| z - z_0 | > | z_1 - z_0 |$. Hence it follows that power series (1) converges either at the point z_0 or at any point z of the complex plane, or there exists a number $R > 0$ such that in a circle $| z - z_0 | < R$ the series converges, and at the points z satisfying the inequality $| z - z_0 | > R$, diverges. The number R is called the radius of convergence, and the circle $| z - z_0 | < R$ the circle of convergence of series (1). If series (1) converges only at the point z_0, then we regard that $R = 0$. And if it converges for any z, then we regard that $R = +\infty$.

In subsection 3 it was shown that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \boldsymbol{R}$.

We define the function e^z of the complex variable z by the formula

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in C. \tag{2}$$

From the preceding note it follows that series (2) converges over the entire complex plane.

Let now $z = iy$, where $y \in \boldsymbol{R}$, be a real number, and i the imaginary unit ($i^2 = -1$). Then from formula (2) we have:

$$e^{iy} = \sum_{n=0}^{\infty} \frac{i^n y^n}{n!} = \sum_{k=0}^{\infty} \frac{i^{2k} y^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} y^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = \cos y + i \sin y,$$

where we have used formulas (5) and (6) of subsection 3. Thus, we have obtained the formula

$$e^{iy} = \cos y + i \sin y. \tag{3}$$

Formula (3) is called Euler's formula.

Example. Compute $e^{2\pi i}$, $e^{-\pi i}$, $e^{\frac{\pi i}{2}}$, $e^{\frac{3\pi i}{2}}$.

△ By Euler's formula (3), we have:

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1;\ e^{-\pi i} = \cos \pi - i \sin \pi = -1;\ e^{\frac{\pi i}{2}}$$

$$= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i;\ e^{\frac{3\pi i}{2}} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i. \ \blacktriangle$$

Exercises

1. Find the radius of convergence of each of the following series:

(a) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$; (b) $\sum\limits_{n=0}^{\infty} \frac{n^2}{2^n} x^n$; (c) $\sum\limits_{n=0}^{\infty} \frac{x^{3n}}{3^n}$;

(d) $\sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$; (e) $\sum\limits_{n=1}^{\infty} n! x^n$; (f) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{3^n} (x-1)^n$.

2. Find the domain of convergence of each of the following series:

(a) $\sum\limits_{n=0}^{\infty} \frac{x^{2n-1}}{2n-1}$; (b) $\sum\limits_{n=1}^{\infty} n^2 x^n$; (c) $\sum\limits_{n=1}^{\infty} \frac{x^n}{n2^n}$;

(d) $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n} x^n$; (e) $\sum\limits_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$; (f) $\sum\limits_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$.

3. Develop each of the given functions as a series in powers of x and find the radius of convergence of the series obtained:

(a) $y = e^{-x^2}$; (b) $y = e^{x^2}$; (c) $y = \cos(x^2)$;

(d) $y = 2^{\frac{x}{2}}$; (e) $y = \ln \frac{4+x}{4-x}$.

4. Derive Taylor's formula with the remainder in Lagrange's form for the functions $y = \frac{1}{1+x^2}$, $x \in R$, and $y = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$. Then write Taylor's series of the indicated functions and determine their domain of convergence.

5. Find Taylor's series for the functions arcsin x and arctan x by integrating Taylor's series for the functions $y = \frac{1}{\sqrt{1-x^2}}$ and $y = \frac{1}{1+x^2}$.

6. With the help of power series, evaluate the following integrals:

(a) $\int_0^A e^{-x^2}\,dx$; (b) $\int_0^A e^{x^2}\,dx$; (c) $\int_0^A \frac{\sin x}{x}\,dx$; (d) $\int_0^A \frac{\arctan x}{x}\,dx$.

7. Compute: $e^{-\pi i}$, $e^{-2\pi i}$, $e^{-\frac{\pi i}{2}}$, $e^{-\frac{3\pi i}{2}}$, $e^{2\pi i n}$, $e^{\frac{\pi i n}{2}}$, $n \in N$.

9.3. Fourier Series

1. Fourier Series for Periodic Functions with Period $T = 2\pi$. In the preceding section, we studied expansions of functions in power series, that is, expansions of composite functions in simpler power functions of the form $a_n (x - x_0)^n$. Such expansions are not always possible and convenient. When studying various periodic processes, it is more natural to expand the functions describing these processes in a series of simple periodic functions—sines and cosines.

Consider the problem on expanding the periodic function $f(x) = 2\pi$ in a trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots + a_n \cos nx + b_n \sin nx + \ldots, \quad (1)$$

where $a_0, a_1, b_1, \ldots, a_n, b_n, \ldots$ are some numbers called the coefficients of the trigonometric series. The term $\frac{a_0}{2}$ is called the absolute term. It is taken in the form $\frac{a_0}{2}$ for convenience.

Suppose that series (1) converges to the function $f(x)$ on the interval $[-\pi, \pi]$, that is, the function $f(x)$ is the sum of series (1):

$$f(x) = \frac{a_0}{2} = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (2)$$

Suppose also that $f(x)$ is a function integrable on the interval $[-\pi, \pi]$ and series (2) can be integrated termwise. Let us find the formulas expressing the coefficients of series (2) in terms of the function $f(x)$.

We intergate equality (2) with respect to x from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0 + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right).$$

Since the integrals of $\cos nx$ and $\sin nx$ between the limits $-\pi$ and π are equal to zero, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{3}$$

We now multiply equality (2) by $\cos kx$, $k \in N$, and then integrate it with respect to x from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right). \tag{4}$$

The integral of $\cos kx$, $k \in N$, between the limits $-\pi$ and π is equal to zero. We evaluate the rest of the integrals using the known trigonometric formulas:

$$2\cos nx\cos kx = \cos(n+k)x + \cos(n-k)x,$$

$$2\sin nx\cos kx = \sin(n+k)x + \sin(n-k)x.$$

For $n \neq k$ we find:

$$\int_{-\pi}^{\pi}\cos nx\cos kx\,dx = \frac{1}{2}\left[\frac{\sin(n+k)x}{n+k} + \frac{\sin(n-k)x}{n-k}\right]\Bigg|_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi}\sin nx\cos kx\,dx = -\frac{1}{2}\left[\frac{\cos(n+k)x}{n+k} + \frac{\cos(n-k)x}{n-k}\right]\Bigg|_{-\pi}^{\pi} = 0.$$

As a result, equality (4) takes the form:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k\int_{-\pi}^{\pi}\cos^2 kx\,dx + b_k\int_{-\pi}^{\pi}\sin kx\cos kx\,dx. \tag{5}$$

Since

$$\int_{-\pi}^{\pi}\sin kx\cos kx\,dx = 0,\quad \int_{-\pi}^{\pi}\cos^2 kx\,dx = \int_{-\pi}^{\pi}\frac{1+\cos 2kx}{2}\,dx = \pi,$$

from equality (5) we find:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \tag{6}$$

for any $k \in N$. It remains to find the formulas expressing the coefficients b_n of series (2). For this purpose, we multiply equality (2) by $\sin kx$, $k \in N$, and then integrate it with respect to x from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin kx \, dx + \sum_{n=1}^{\infty}{}' \left(a_n \int_{-\pi}^{\pi} \cos nx \sin kx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin kx \, dx \right).$$

Repeating the computations carried out earlier, we find

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k \in N. \tag{7}$$

Note that for any function, integrable on the interval $[-\pi, \pi]$, we can find the numbers a_0, a_k, and b_k by formulas (3), (6), and (7), and therefore for such a function it is possible to write trigonometric series (1) which, generally speaking, can either converge or diverge.

Definition. Trigonometric series (1) whose coefficients are computed by formulas (3), (6), and (7) is called the Fourier series of the

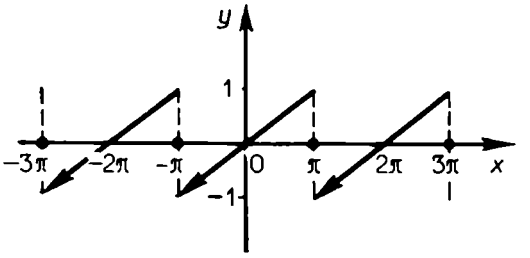

Fig. 128

function $f(x)$, and the coefficients a_0, a_n, b_n are called the Fourier coefficients of the function $f(x)$.

It follows from the definition that to find the Fourier series of a given function means to find the Fourier coefficients of this function by formulas (3), (6), and (7) and to write trigonometric series (1) with these coefficients.

Example 1. Find the Fourier series for a periodic function $f(x)$ with period 2π given on the interval $(-\pi, \pi)$ by the formula $f(x) = x$ (Fig. 128).

△ We compute the Fourier coefficients by formulas (3), (6), and (7)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{x^2}{2\pi}\Big|_{-\pi}^{\pi} = 0;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \frac{1}{\pi n}\int_{-\pi}^{\pi} x\,d\sin nx$$

$$= \frac{x\sin nx}{\pi n}\Big|_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi}\sin nx\,dx = \frac{\cos nx}{\pi n^2}\Big|_{-\pi}^{\pi} = 0;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = -\frac{1}{\pi n}\int_{-\pi}^{\pi} x\,d\cos nx$$

$$= -\frac{x\cos nx}{\pi n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi}\cos nx\,dx = -\frac{\cos n\pi}{n}$$

$$-\frac{\cos n\pi}{n} = (-1)^{n+1}\cdot\frac{2}{n}.$$

Thus, we get the Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}.$$

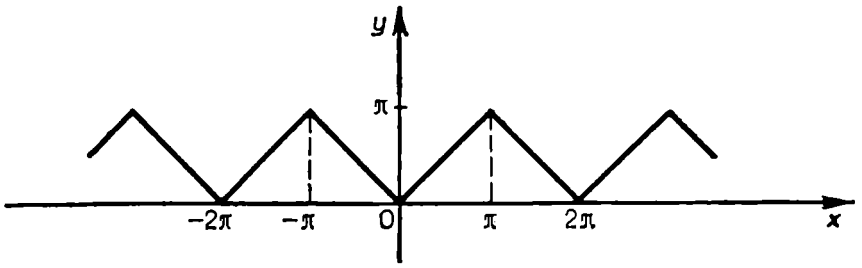

Fig. 129

To show that this Fourier series corresponds to the function $f(x)$, we shall write:

$$f(x) \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}. \quad ▲$$

Example 2. Find the Fourier series for a periodic function $f(x)$ with period 2π given on the interval $[-\pi, \pi]$ by the formula $f(x) = |x|$ (Fig. 129).

△ We compute the Fourier coefficients by formulas (3), (6), and (7):

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\,dx+\int_{0}^{\pi} x\,dx\right]=\pi;$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx+\int_{0}^{\pi} x\cos nx\,dx\right]$$

$$=\frac{1}{\pi n}\left[-\int_{-\pi}^{0} x\,d\sin nx+\int_{0}^{\pi} x\,d\sin nx\right]$$

$$=\frac{1}{\pi n}\left[-x\sin nx\Big|_{-\pi}^{0}+\int_{-\pi}^{0}\sin nx\,dx+x\sin nx\Big|_{0}^{\pi}-\int_{0}^{\pi}\sin nx\,dx\right]$$

$$=\frac{1}{\pi n^2}\left[-\cos nx\Big|_{-\pi}^{0}+\cos nx\Big|_{0}^{\pi}\right]=\frac{1}{\pi n^2}\left[-2+2\cos \pi n\right]$$

$$=\frac{2\left[(-1)^n-1\right]}{\pi n^2}.$$

Computing the coefficients b_n in a similar way, we find:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx+\int_{0}^{\pi} x\sin nx\,dx\right]=0.$$

Thus,

$$f(x)\sim\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]\cos nx}{n^2}.$$ ▲

2. Fourier Series for Even and Odd Functions. The examples considered in the preceding subsection show that for an odd function (Example 1) the Fourier series contains only terms with sin nx, while for an even function (Example 2) only the absolute term and terms with cos nx. Let us prove this assertion in the general form.

Lemma 1. *If an integrable function* $f(x)$, $x\in[-l,\ l]$, *is even, then*

$$\int_{-l}^{l} f(x)\,dx=2\int_{0}^{l} f(x)\,dx, \tag{1}$$

and if odd, then

$$\int_{-l}^{l} f(x)\,dx=0. \tag{2}$$

□ We have:

$$\int_{-l}^{l} f(x)\,dx = \int_{0}^{l} f(x)\,dx + \int_{-l}^{0} f(x)\,dx.$$

We made the substitution $x = -u$ in the last integral. Then

$$\int_{-l}^{0} f(x)\,dx = -\int_{l}^{0} f(-u)\,du = \int_{0}^{l} f(-u)\,du,$$

and therefore

$$\int_{-l}^{l} f(x)\,dx = \int_{0}^{l} f(x)\,dx + \int_{0}^{l} f(-x)\,dx \tag{3}$$

(the variable of integration in the last integral is denoted by the letter x once again). Formula (3) implies that if $f(x)$ is even, then equality (1) is valid, and if it is odd, then equality (2) is valid. ■

By virtue of the proved lemma, the Fourier coefficients for a periodic function $f(x)$ with period 2π are computed by the formulas

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \quad b_n = 0, \tag{4}$$

since for any $n \in N$ the function $f(x)\cos nx$ is even, and the function $f(x)\sin nx$ is odd. The Fourier series of an even function $f(x)$ contains no terms with sines:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

and the coefficients a_0, a_n are computed by formula (4).

For an odd function $f(x)$ we have:

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx, \tag{5}$$

since the function $f(x)\cos nx$ is odd, and the function $f(x)\sin nx$ is even. Thus, the Fourier series of an odd function contains no absolute term and terms with cosines:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx,$$

and the coefficients b_n are computed by formula (5).

Example 1. Find the Fourier series for the periodic function $f(x)$ with period 2π (Fig. 130):

$$f(x)=\begin{cases}-1 \text{ for } -\pi<x<0,\\ 0 \text{ for } x=-\pi,\ x=0 \text{ and } x=\pi,\\ 1 \text{ for } 0<x<\pi.\end{cases}$$

△ Since the given function is odd, we have $a_0 = a_n = 0$. We compute b_n by formula (5):

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$=-\frac{2}{\pi n}\cos nx\Big|_0^{\pi}=\frac{2}{\pi n}[1-(-1)^n].$$

Consequently,

$$f(x)\sim\frac{2}{\pi}\sum_{n=1}^{\infty}[1-(-1^n)]\frac{\sin nx}{n}. \ \blacktriangle$$

Note that it is sufficient to define any periodic function with period 2π on some half-interval of length 2π. At the rest of the points it

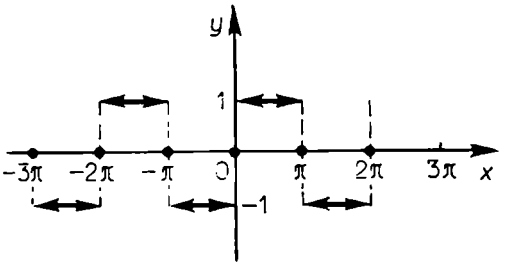

Fig. 130

will be defined by virtue of periodicity. Let us find the formulas expressing the Fourier coefficients of this function in terms of its values on an arbitrary interval of length 2π. But first we shall prove the following assertion:

Lemma 2. If $f(x)$ is a periodic function with period $T>0$, then

$$\int_0^T f(x)\,dx=\int_a^{a+T} f(x)\,dx$$

for any $a\in\boldsymbol{R}$.

□ We have:

$$\int_0^T f(x)\,dx=\int_0^a f(x)\,dx+\int_a^{a+T} f(x)\,dx+\int_{a+T}^T f(x)\,dx.$$

We transform the last integral:

$$\int_{a+T}^{T} f(x)\,dx = -\int_{T}^{a+T} f(x)\,dx = -\int_{0}^{a} f(u+T)\,du$$

$$= -\int_{0}^{a} f(u)\,du = -\int_{0}^{a} f(x)\,dx,$$

where we have introduced a new variable of integration $u = x - T$ and have denoted the variable of integration u in the last integral again by the letter x. Hence it follows that the integral from 0 to T is equal to the integral from a to $a + T$. ■

According to Lemma 2, from formulas (3), (6), and (7) in the preceding subsection we find:

$$a_0 = \frac{1}{\pi}\int_{a}^{a+2\pi} f(x)\,dx, \tag{6}$$

$$a_n = \frac{1}{\pi}\int_{a}^{a+2\pi} f(x)\cos nx\,dx, \tag{7}$$

$$b_n = \frac{1}{\pi}\int_{a}^{a+2\pi} f(x)\sin nx\,dx \tag{8}$$

for any $a \in \boldsymbol{R}$ since $f(x)\cos nx$ and $f(x)\sin nx$ are periodic functions with period 2π.

Example 2. Find the Fourier series for a periodic function $f(x)$ with period 2π given by the formula $f(x) = x$ on the half-interval $x \in (0,\ 2\pi]$.

△ On the closed interval $[-\pi,\ \pi]$ this function is given by two formulas: $f(x) = x + 2\pi$, $x \in [-\pi,\ 0]$, and $f(x) = x$, $x \in (0,\ \pi)$. At the same time, on the half-interval $(0,\ 2\pi]$ it is given by one formula $f(x) = x$. Therefore it is more convenient to use formulas (6), (7), and (8) for computing the Fourier coefficients. Setting $a = 0$, we compute:

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} x\,dx = 2\pi;$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} x\cos nx\,dx = \frac{1}{\pi n}\int_{0}^{2\pi} x\,d\sin nx$$

$$= \frac{x\sin nx}{\pi n}\Big|_0^{2\pi} - \frac{1}{\pi n}\int_{0}^{2\pi}\sin nx\,dx = \frac{\cos nx}{\pi n^2}\Big|_0^{2\pi} = 0;$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx\,dx = -\frac{1}{\pi n}\int_0^{2\pi} x\,d\cos nx$$
$$= -\frac{2\cos 2\pi n}{n} = -\frac{2}{n}.$$

Consequently,

$$f(x) \sim \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}. \quad \blacktriangle$$

3. The Convergence of Fourier Series. A function $f(x)$ is said to be piecewise monotonic on some interval if the latter can be divided into a finite number of subintervals on each of which it is monotonic. We give without proof the following theorem on the convergence of the Fourier series:

Theorem 9.3.1. *If a periodic function $f(x)$ with period 2π is bounded and piecewise monotonic on the interval $[a, a + 2\pi]$, $a \in \boldsymbol{R}$, then the Fourier series of this function converges at all points $x \in \boldsymbol{R}$, converging to $f(x)$ at points of continuity of the function $f(x)$ and to $\frac{f(x-0)+f(x+0)}{2}$ at points of discontinuity.*

In other words, a periodic function (with period 2π), piecewise monotonic on the interval $[a, a + 2\pi]$, $a \in \boldsymbol{R}$, can be expanded in a Fourier series at all points of continuity.

For the continuous function $f_1(x) = |x|$, $x \in [-\pi, \pi]$, $f_1(x + 2\pi) = f_1(x)$ considered in Example 2 of subsection 1 the Fourier series converges to $f_1(x)$ for all x:

$$f_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right). \qquad (1)$$

For the discontinuous function $f_2(x)$ considered in Example 1 of subsection 2 the equality $f_2(x) = \frac{f_2(x-0)+f_2(x+0)}{2}$ is valid throughout the real axis. Consequently, the Fourier series converges to $f_2(x)$ for all x:

$$f_2(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right). \qquad (2)$$

Let us calculate the partial sums of Fourier series (1) on a personal computer. For this purpose, we write the program in BASIC:

```
1 INPUT X, N
2 LET S = 0
3 LET M = 1
4 FOR I = 1 TO N
```

```
 5 LET S = S + COS (M * X)/(M * M)
 6 LET M = M + 2
 7 NEXT 1
 8 LET P = 4 * ATN (1)
 9 LET F = P/2 - 4 * S/P
10 PRINT "F = "; F
11 END
```

Here N is the number of terms. In line 8 we compute the constant $\pi = 4 \arctan(1)$, and the function arctan x in BASIC is denoted by ATN (X).

The program for computing the partial sums of Fourier series (2) is quite similar:

```
 1 INPUT X, N
 2 LET S = 0
 3 LET M = 1
 4 FOR I = 1 TO N
 5 LET S = S + SIN (M * X)/M
 6 LET M = M + 2
 7 NEXT 1
 8 LET P = 4 * ATN (1)
 9 LET F = 4 * S/P
10 PRINT "F = "; F
11 END
```

Figure 131 shows the graphs of the function $f_1(x)$ and the partial sum of Fourier series (1) for N = 3, and Figure 132 and 133 the graphs of the function $f_2(x)$ and the partial sums of Fourier series (2) for N = 3 and N = 8. We see that the deviation of the partial sums of the Fourier series from the values of the function near the points of discontinuity does not tend to zero as $N \to \infty$. However, the convergence of the Fourier series for these points is ensured by the fact that the length of the interval on which the deviation is great tends to zero as $N \to \infty$.

4. Fourier Series for Periodic Functions with Period $T = 2l$. Any periodic function $f(x)$ with period $T = 2l$, $l > 0$, is transformed into the function $\varphi(t) = f\left(\frac{lt}{\pi}\right)$ with period 2π by the substitution

$$x = \frac{l}{\pi} t. \tag{1}$$

Indeed,

$$\varphi(t + 2\pi) = f\left(\frac{l(t+2\pi)}{\pi}\right) = f\left(\frac{lt}{\pi} + 2l\right) = f\left(\frac{lt}{\pi}\right) = \varphi(t).$$

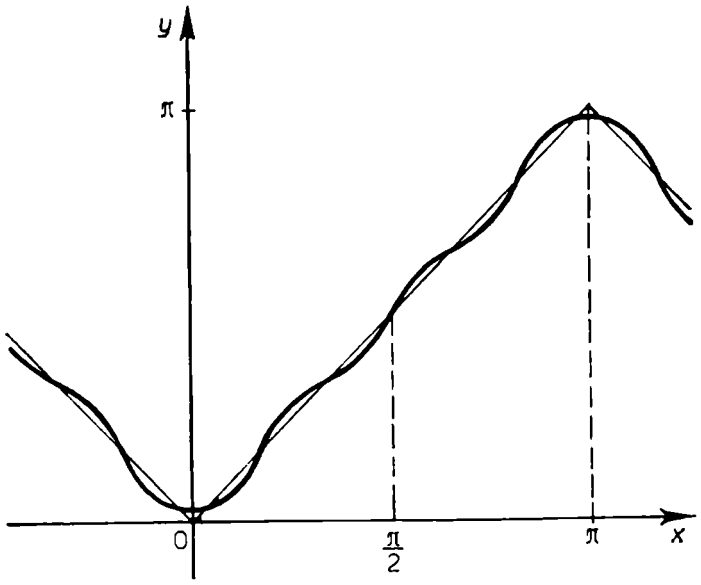

Fig. 131

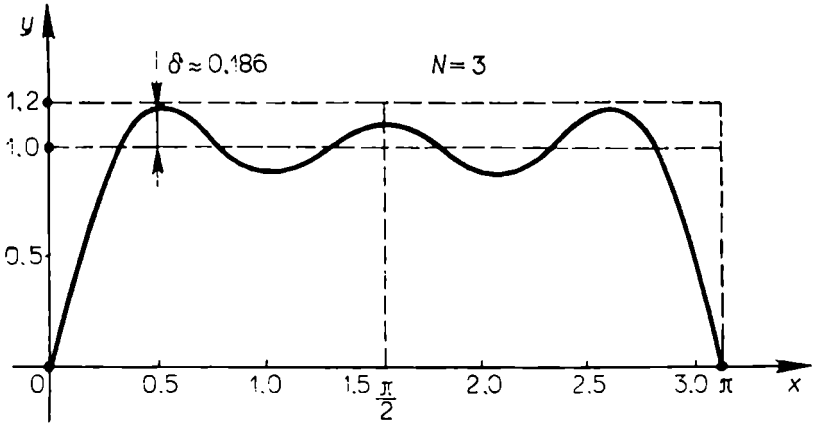

Fig. 132

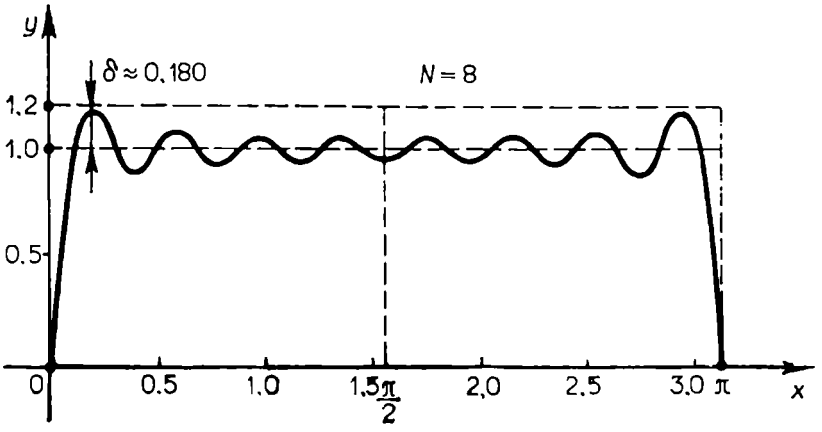

Fig. 133

If the function $f(x)$ is given on the interval $[a,\ a+2l]$, $l>0$, then the function $\varphi(t)$ will be given on the interval $\left[\frac{\pi a}{l}; \frac{\pi a}{l}+2\pi\right]$. We write the Fourier series for the function $\varphi(t)$:

$$\varphi(t) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Hence, making the substitution $t=\frac{\pi x}{l}$, inverse to substitution (1), we get the corresponding series for the function $f(x)$:

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos \frac{\pi n x}{l}+b_n \sin \frac{\pi n x}{l}\right). \tag{2}$$

This series is called the Fourier series of the periodic function $f(x)$ with period $2l$. We then find the corresponding formulas for the coefficients of Fourier series (2). Note that the series for the functions $\varphi(t)$ and $f(x)$ have the same coefficients. For the function $\varphi(t)$ they are computed by formulas (6), (7), and (8) from subsection 2:

$$a_0=\frac{1}{\pi}\int_c^{c+2\pi}\varphi(t)\,dt,$$

$$a_n=\frac{1}{\pi}\int_c^{c+2\pi}\varphi(t)\cos nt\,dt, \quad b_n=\frac{1}{\pi}\int_c^{c+2\pi}\varphi(t)\sin nt\,dt,$$

where c is set to be equal to $\frac{\pi a}{l}$.

Passing over in these formulas to the variable $x=\frac{lt}{\pi}$, we find:

$$a_0=\frac{1}{l}\int_a^{a+2l} f(x)\,dx, \tag{3}$$

$$a_n=\frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{\pi n x}{l}\,dx, \tag{4}$$

$$b_n=\frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{\pi n x}{l}\,dx. \tag{5}$$

The theorem on the convergence of the Fourier series for a periodic function with period 2π yields the corresponding theorem on the convergence of the Fourier series with an arbitrary period.

Theorem 9.3.2. *If a $2l$-periodic function $f(x)$ is bounded and piecewise monotonic on the interval $[a,\ a+2l]$, $a\in\boldsymbol{R}$, $l>0$, then its*

Fourier series converges to $\frac{f(x-0)+f(x+0)}{2}$ *at any point* $x \in \boldsymbol{R}$. *In particular, at points of continuity of the function the series converges to* $f(x)$.

Example 1. Expand in a Fourier series the periodic function $f(x)$ with period $T = 2$ given on the half-interval $[-1, 1]$ by the formula

$$f(x) = \begin{cases} 0 \text{ if } x \in [-1, 0), \\ 1 \text{ if } x \in [0, 1). \end{cases}$$

△ For the given function $a = -1$ and $l = 1$. By formulas (3), (4), and (5), we find:

$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{0}^{1} dx = 1; \quad a_n = \int_{-1}^{1} f(x)\cos \pi n x\,dx = \int_{0}^{1} \cos \pi n x\,dx$$

$$= \frac{\sin \pi n x}{\pi n}\Big|_0^1 = \frac{\sin \pi n}{\pi n} = 0; \quad b_n = \int_{-1}^{1} f(x)\sin \pi n x\,dx = \int_{0}^{1} \sin \pi n x\,dx$$

$$= -\frac{\cos \pi n x}{\pi n}\Big|_0^1 = \frac{1-\cos \pi n}{\pi n} = \frac{1-(-1)^n}{\pi n}.$$

The equality

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n} \sin \pi n x$$

is valid for all x from the half-interval $[-1, 1)$ except the points of discontinuity of the function $x = -1$ and $x = 0$ at which the Fourier series converges to 2. ▲

If the function $f(x)$, $x \in [-l, l]$, $l > 0$, is even, then, obviously, the function $f(x)\cos\frac{\pi n x}{l}$ is even, and the function $f(x)\sin\frac{\pi n x}{l}$ is odd, and therefore (see Lemma 1 in subsection 2)

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{\pi n x}{l}\,dx, \quad b_n = 0, \quad n \in \boldsymbol{N}. \quad (6)$$

Similarly, if the function $f(x)$, $x \in [-l, l]$, is odd, then the function $f(x)\cos\frac{\pi n x}{l}$ is odd, and the function $f(x)\sin\frac{\pi n x}{l}$ is

even, and therefore

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}\,dx, \quad n \in N. \tag{7}$$

Example 2. Expand in a Fourier series the periodic function $f(x)$ with period $T = 2$ given on the open interval $(-1, 1)$ by the formula $f(x) = x$ and equal to zero at the points $x = \pm 1$.

△ The given function is odd. Setting $l = 1$ in formulas (7), we find $a_0 = 0$, $a_n = 0$, and

$$b_n = 2\int_0^1 x\sin\pi n x\,dx = -\frac{2}{\pi n}\int_0^1 x\,d\cos\pi n x$$

$$= -\frac{2x\cos\pi n x}{\pi n}\Big|_0^1 + \frac{2}{\pi n}\int_0^1 \cos\pi n x\,dx = -\frac{2\cos\pi n}{\pi n} = \frac{2(-1)^{n+1}}{\pi n}.$$

The theorem on the convergence of Fourier series implies that

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\pi n x, \quad x \in [-1, 1]. \quad \blacktriangle$$

5. Expanding Nonperiodic Functions in a Fourier Series. Let on the half-interval $[a, b)$ a bounded piecewise monotonic function $f(x)$ be given. Let us extend this function throughout the entire real axis in a periodic manner. The function thus obtained can be expanded in a Fourier series which will converge to $\frac{f(x-0)+f(x+0)}{2}$ at each point of the half-interval $[a, b)$.

If the function $f(x)$ is given on the closed interval $[0, l]$, then it is convenient first to define it on the interval $[-l, 0]$ by the formula $f(x) = f(-x)$ (or $f(x) = -f(-x)$) as a result of which we get an even (odd) function. On being periodically extended to the entire number axis (with a possible change in the values of the function at discontinuities), this function will become periodic with period $2l$. Such a function is developed as a Fourier series only in cosines (for an odd function—in sines).

Example 1. Develop the function $f(x) = \sin x$, $x \in [0, \pi]$ as a Fourier series in cosines.

△ We extend the given function to the interval $[-\pi, 0]$ in an even manner and spread it periodically with period 2π to the entire real axis. Thereby, we get the function $f(x) = |\sin x|$, $x \in R$. Since this function is even, $b_n = 0$, $n \in N$. We compute the coeffici-

ents a_0, a_n by formula (6) in the preceding subsection ($l = \pi$):

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx = \frac{4}{\pi}; \quad a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = 0;$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{2}{\pi}\,\frac{(-1)^{n-1}-1}{n^2-1}, \quad n \geqslant 2.$$

The theorem on the convergence of a Fourier series implies that

$$|\sin x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^{n-1}-1}{n^2-1}\cos nx, \quad x \in R.$$

On the closed interval $[0, \pi]$ the found series converges to $\sin x$. ▲

Example 2. Expand the function $f(x) = \cos x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

△ Extending the given function periodically to the entire number axis, we get an even periodic function with period π whose Fourier coefficients are computed by formula (6) in the preceding subsection $\left(l = \frac{\pi}{2}\right)$:

$$b_n = 0, \ n \in N; \quad a_0 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x\, dx = \frac{4\sin x}{\pi}\Big|_0^{\frac{\pi}{2}} = \frac{4}{\pi};$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos 2nx\, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} [\cos(2n+1)x + \cos(2n-1)x]\, dx$$

$$= \frac{2\sin(2n+1)x}{\pi(2n+1)}\Big|_0^{\frac{\pi}{2}} + \frac{2\sin(2n-1)x}{\pi(2n-1)}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[\frac{\sin\left(\pi n + \frac{\pi}{2}\right)}{2n+1} + \frac{\sin\left(\pi n - \frac{\pi}{2}\right)}{2n-1}\right]$$

$$= \frac{2(-1)^n}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{4}{\pi}\,\frac{(-1)^{n+1}}{4n^2-1}.$$

As a result of periodical extension of the given function to the entire number axis, we get the function $|\cos x|$. The Fourier series con-

verges to $|\cos x|$ throughout the number axis:

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} \cos 2nx, \quad x \in R.$$

On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the found series converges to $\cos x$. ▲

6. The Complex Form of Fourier Series. Let us write the Fourier series for a periodic function $f(x)$ (with period $2l$) given on the closed interval $[a, a + 2l]$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{l} + b_n \sin \frac{\pi n x}{l}\right).$$

From Euler's formula (see 9.2, subsection 5)

$$e^{iy} = \cos y + i \sin y$$

and the implied equality

$$e^{-iy} = \cos y - i \sin y,$$

we find that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin y = \frac{e^{iy} - e^{-iy}}{2t} \tag{2}$$

for any $y \in R$.

We transform the Nth partial sum of series (1) with the help of formula (2):

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{\pi n x}{l} + b_n \sin \frac{\pi n x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{\frac{i\pi n x}{l}} + e^{\frac{-i\pi n x}{l}}}{2} + b_n \frac{e^{i\frac{\pi n x}{l}} - e^{-i\frac{\pi n x}{l}}}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} \frac{a_n - ib_n}{2} e^{i\frac{\pi n x}{l}} + \sum_{n=1}^{N} \frac{a_n + ib_n}{2} e^{-i\frac{\pi n x}{l}}$$

$$= \sum_{m=-N}^{N} c_m e^{i\frac{\pi m x}{l}},$$

where we have used the following designation:

$$c_0 = \frac{a_0}{2}, \quad c_m = \frac{a_m - ib_m}{2} \quad \text{for } m > 0,$$

$$c_m = \frac{a_{-m} + ib_{-m}}{2} \quad \text{for } m < 0,$$

the summation being made over all integral m and from $-N$ to N.

The theorem on the convergence of a Fourier series implies that for the functions bounded and piecewise monotonic on the closed interval $[a, a+2l]$ the limit of partial sums of the Fourier series, as $N \to \infty$, exists, and, consequently, the limit

$$\lim_{N\to\infty} \sum_{m=-N}^{N} c_m e^{i\frac{\pi m x}{l}}$$

exists. This limit is denoted by

$$\sum_{m=-\infty}^{+\infty} c_m e^{i\frac{\pi m x}{l}}$$

and is called the Fourier series in complex form.

Let us find the expressions for the complex coefficients c_m in terms of the function $f(x)$. If $m = 0$, then

$$c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx.$$

If $m > 0$, then

$$c_m = \frac{a_m - ib_m}{2} = \frac{1}{2l}\int_a^{a+2l} f(x)\left(\cos\frac{\pi m x}{l} - i\sin\frac{\pi m x}{l}\right)dx$$

$$= \frac{1}{2l}\int_a^{a+2l} f(x)\,e^{-i\frac{\pi m x}{l}}\,dx.$$

If $m < 0$, then

$$c_m = \frac{a_{-m} + ib_{-m}}{2} = \frac{1}{2l}\int_a^{a+2l} f(x)\left(\cos\frac{\pi m x}{l} - i\sin\frac{\pi m x}{l}\right)dx$$

$$= \frac{1}{2l}\int_a^{a+2l} f(x)\,e^{-i\frac{\pi m x}{l}}\,dx,$$

since $\cos\frac{\pi(-m)x}{l} = \cos\frac{\pi m x}{l}$ and $\sin\frac{\pi(-m)x}{l} = -\sin\frac{\pi m x}{l}$. Thus, for any integral m the following formula is valid:

$$c_m = \frac{1}{2l}\int_a^{a+2l} f(x)\,e^{-i\frac{\pi m x}{l}}\,dx. \tag{3}$$

Example. Write the Fourier series in complex form for a periodic (with period $T = 2$) function given on the half-interval $[-1, 1)$

by the equalities

$$f(x)=\begin{cases}0 \text{ if } x\in[-1,\ 0),\\ 1 \text{ if } x\in[0,\ 1).\end{cases}$$

△ We compute the complex coefficients of the Fourier series by formula (3), taking into consideration that for the given function $a=-1$ and $l=\frac{T}{2}=1$:

$$c_0=\frac{1}{2}\int_{-1}^{1} f(x)\,dx=\frac{1}{2}\int_{0}^{1} dx=\frac{1}{2},$$

$$c_m=\frac{1}{2}\int_{-1}^{1} f(x)\,e^{-i\pi mx}\,dx=\frac{1}{2}\int_{0}^{1} e^{-i\pi mx}\,dx$$

$$=\frac{i}{2\pi m}\left(e^{-i\pi m}-1\right)=\frac{i}{2\pi m}\left((-1)^m-1\right)$$

for $m\neq 0$. Consequently,

$$f(x)\sim\frac{1}{2}+\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\frac{i\left((-1)^m-1\right)}{2\pi m}\,e^{i\pi mx}. \quad ▲$$

Exercises

1. Find the Fourier series of the following periodic functions with period 2π:

(a) $f(x)=\begin{cases}2 \text{ if } x\in(-\pi,\ 0),\\ 5 \text{ if } x\in[0,\ \pi];\end{cases}$ (b) $f(x)=\cos^2 x,\ x\in[-\pi,\ \pi]$;

(c) $f(x)=\begin{cases}2x \text{ if } x\in[-\pi,\ 0],\\ 0 \text{ if } x\in[0,\ \pi).\end{cases}$

2. Expand the following periodic functions ($T=2\pi$) in a Fourier series:

(a) $f(x)=\sin\frac{x}{2}$, $x\in[-\pi,\ \pi]$; (b) $f(x)=(\sin x+\cos x)^2$, $x\in[-\pi,\ \pi]$;

(c) $f(x)=x^2$, $x\in[-\pi,\ \pi]$.

3. Develop the following functions as a Fourier series in cosines:

(a) $f(x)=\begin{cases}1 \text{ if } x\in[0,\ 1],\\ 0 \text{ if } x\in(1,\ \pi);\end{cases}$

(b) $f(x)=\begin{cases}1-\frac{x}{2} \text{ if } x\in[0,\ 2),\\ 0 \text{ if } x\in[2,\ \pi);\end{cases}$

(c) $f(x)=3x^2-6\pi x+2\pi^2$, $x\in[0,\ \pi)$.

4. Develop the following functions as a Fourier series in sines:

$$\text{(a) } f(x) = \cos x,\ x \in (0,\ \pi);$$

$$\text{(b) } f(x) = \sin \frac{x}{2},\ x \in [0,\ \pi);$$

$$\text{(c) } f(x) = x(\pi - x),\quad x \in [0,\ \pi].$$

5. Expand the following periodic functions ($T = 2\pi$) in a Fourier series:

$$\text{(a) } f(x) = \begin{cases} \frac{\pi - x}{2} & \text{if } x \in (0,\ 2\pi), \\ 0 & \text{if } x = 0; \end{cases}$$

$$\text{(b) } f(x) = \begin{cases} x & \text{if } x \in (1,\ 1 + 2\pi), \\ 1 + \pi & \text{if } x = 1. \end{cases}$$

6. Expand the following functions in a Fourier series (sketch the graph of the sum of the obtained series):

$$\text{(a) } f(x) = |x|,\ x \in [-1,\ 1];\quad \text{(b) } f(x) = \begin{cases} x & \text{if } x \in [0,\ 1), \\ 2 - x & \text{if } x \in [1,\ 2]; \end{cases}$$

$$\text{(c) } f(x) = \begin{cases} A & \text{if } x \in [0,\ l); \\ 0 & \text{if } x \in [l,\ 2l); \end{cases}\quad \text{(d) } f(x) = \begin{cases} x & \text{if } x \in [0,\ 1), \\ 1 & \text{if } x \in [1,\ 2], \\ 3 - x & \text{if } x \in [2,\ 3]. \end{cases}$$

7. Develop the function

$$f(x) = \begin{cases} x & \text{if } x \in [0,\ 1), \\ 2 - x & \text{if } x \in [1,\ 2] \end{cases}$$

as a Fourier series in sines.

8. Expand the periodic ($T = 1$) function

$$f(x) = \begin{cases} 1 - x & \text{if } x \in (0,\ 1), \\ \frac{1}{2} & \text{if } x = 0, \end{cases}$$

in a Fourier series.

9. Expand the periodic ($T = 10$) function

$$f(x) = \begin{cases} 10 - x & \text{if } x \in (5,\ 15), \\ 0 & \text{if } x = 5, \end{cases}$$

in a Fourier series.

10. Expand the periodic ($T = 4$) function

$$f(x) = \begin{cases} x \text{ if } x \in [0, 1), \\ 1 \text{ if } x \in [1, 2), \\ 3 - x \text{ if } x \in [2, 3), \\ 0 \text{ if } x \in [3, 4), \end{cases}$$

in a Fourier series.

11. Expand in a Fourier series (in complex form) the periodic ($T = \pi$) function given on the half-interval $(0, \pi]$ by the equalities

$$f(x) = \begin{cases} \cos x \text{ if } x \in \left(0, \frac{\pi}{2}\right), \\ 0 \text{ if } x \in \left[\frac{\pi}{2}; \pi\right), \\ \frac{1}{2} \text{ if } x = \pi. \end{cases}$$

Chapter 10

DIFFERENTIAL CALCULUS: FUNCTIONS OF SEVERAL VARIABLES

10.1. Functions of Several Variables

1. Problems Leading to the Notion of a Function of Several Variables. As is known, if x and y denote the lengths of sides of a rectangle, and S denotes its area, then

$$S = x \cdot y. \tag{1}$$

The area S varies with a change in x and y. In this case, the area S is said to be a function of two variables x and y given by formula (1). Similarly, if x, y, and z denote the lengths of edges of a rectangular parallelepiped, then its volume V is computed by formula

$$V = x \cdot y \cdot z. \tag{2}$$

Changing x, y, and z, we shall obtain, by formula (2), various values for V, that is, V is a function of three variables x, y, and z given by formula (2).

When studying the properties of a heated body, its temperature is usually a variable quantity which depends both on the point at which temperature is measured and instant of time at which the measurement is carried out.

Let T denote the temperature being measured, M the point at which temperature is measured, and t the instant of time at which it is measured.

Then the dependence of temperature T on a variable point M and time t is denoted as follows:

$$T = f(M, t).$$

In this case, we say that temperature is a function of the point M and time t.

Let the point M have coordinates (x, y, z). Then the dependence of the temperature on the coordinates of the point and the instant of time t is denoted in the following way:

$$T = f(x, y, z, t),$$

that is, the temperature is a function of four variables x, y, z, and t. The variables x, y, z, and t are independent variables, they can take on any permissible values.

The variable T is a dependent variable whose values are determined by the values of the independent variables x, y, z, and t.

2. The Definition of a Function of Several Variables. As we already know, the notion of a function involves the establishment of a dependence (relation) between the members of two sets. Let G be a set of points in the xy-plane. If to every point $M(x, y)$ of the set G there corresponds, by the rule f, a unique number u from the set B, then we say that a function of the point $f(M)$ is defined on the set G and write

$$u = f(M), \quad M \in G. \tag{1}$$

The set G is called the domain of definition of the function, and the set B is termed the range.

Since every point M is uniquely characterized by its Cartesian coordinates (x, y), instead of a function of a point, we may speak of a function of two variables x and y:

$$u = f(x, y), \quad (x, y) \in G. \tag{2}$$

For example, $f(x, y) = x^2 + y^2$, $(x, y) \in R^2$. This function places each point in the xy-plane having coordinates (x, y) in correspondence with a certain number $u = x^2 + y^2$.

The variables x and y are called the independent variables or arguments of the function, while the variable u is termed the dependent variable or value of the function.

Functions of two variables are often given by formulas. The set of points M for which the formula has sense is called the natural domain of definition of a function given by a formula. If a function given by a formula is considered on the natural domain of definition, then, when describing the function, we indicate only the formula. For instance, the formula $z = \sqrt{1 - x^2 - y^2}$ specifies the function of two variables defined for all points (x, y) from the unit circle

$$x^2 + y^2 \leqslant 1.$$

In a three-dimensional space, the coordinates of points are represented by ordered triples of numbers (x, y, z). In this case, the function of the point $f(M)$ is a function of three variables x, y, and z. For example,

$$u = \sqrt{x + y + z - 1}.$$

In the plane and in a three-dimensional space, points are characterized by ordered collections of two or three numbers, respectively. The distance $|MM_0|$ between the points M and M_0 in the plane is computed by the formula

$$|MM_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

and in the case of a three-dimensional space by the formula

$$|MM_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

Let us now define a point of an abstract n-dimensional space as an ordered collection of n real numbers $(x_1, x_2, \ldots, x_n)$. If the distance between the points $M(x_1, x_2, \ldots, x_n)$ and $M_0(y_1, y_2, \ldots, y_n)$ of an n-dimensional space is determined by the equality

$$|MM_0| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2},$$

then the so defined space is called the n-dimensional space of points and is denoted by R^n.

The function $f(M)$ of the point $M(x_1, \ldots, x_n)$ of the n-dimensional space is called a function of n variables $x_1, x_2, \ldots, x_n$. Functions of n variables can be defined by formulas. For instance,

$$u = \sqrt{x_1^2+x_2^2+\ldots+x_n^2+3}.$$

This function is defined everywhere in n-dimensional space, that is, for any values of real variables $x_1, x_2, \ldots, x_n$.

For a complete characteristic of sets of points in n-dimensional spaces (for $n = 1$ and $n = 2$ we get an ordinary plane and a three-dimensional space, respectively) we introduce the notions of boundedness, connectivity, openness, closure, and, finally, the notion of open and closed domains.

Let us explain these notions by considering examples of sets of points in the plane.

A set of points in the plane is said to be bounded if there is a square (or a circle) enclosing all the points of this set.

An open domain is a point set possessing the following properties:

1. Every point of the set $M_0(x_0, y_0)$ has a neighbourhood, that is, an open circle (a circle without circumference) with centre at the point $M_0(x_0, y_0)$ all the points of which belong to the given set.
2. Any two points of the set can be connected with a polygonal line whose points belong to the given set.

The first property is called the openness of a set, the second property characterizes its connectivity.

A boundary point of a set is defined as a point in any neighbourhood of which there are points both belonging and not belonging to this set. The totality of all boundary points is called the boundary of this set. For example, the boundary of a circle is its circumference.

A closed domain is defined as a set consisting of all the points of an open domain and its boundary. For example, a closed circle is a circle together with its circumference.

Example 1. Find the natural domain of definition of the function given by the formula

$$u = \frac{1}{\sqrt{1-(x^2+y^2)}}.$$

△ The natural domain of definition is a set of points for which this formula has sense. In this case, the radicand must be neither

negative (since there is no root) nor zero (division by zero is impossible), that is, the following inequality must be satisfied:

$$x^2 + y^2 < 1.$$

This inequality is satisfied by the points of an open circle (without circumference) with centre at the origin and radius equal to unity. ▲

Example 2. Find the natural domain of definition of the function given by the formula

$$u = \sqrt{1-(x^2+y^2)} + \sqrt[4]{x^2+y^2-1}.$$

△ Since the inequalities $x^2 + y^2 \leqslant 1$ and $x^2 + y^2 \geqslant 1$ must be satisfied simultaneously, the domain of definition of the given function is a line, namely, a circle with centre at the origin and radius equal to unity. It is of interest to note that the set of values (the range) of the given function consists of one number zero. ▲

The set of points in space with coordinates $(x, y, f(x, y))$, where $(x, y) \in G$ is called the graph of a function of two variables $z = f(x, y)$. For example, the graph of the function $z = x^2 + y^2$ is a paraboloid of revolution shown in Fig. 50.

3. The Limit and Continuity of a Function of Two Variables. Recall that any open circle (that is, a circle without its circumference) centered at the point $M_0(x_0, y_0)$ is called the neighbourhood of this point.

If the radius of this circle is equal to δ, then this neighbourhood is called the δ-neighbourhood of the point $M_0(x_0, y_0)$. Consequently, the δ-neighbourhood of the point $M_0(x_0, y_0)$ is the set of all points M with coordinates (x, y) satisfying the inequality

$$|MM_0|^2 = (x - x_0)^2 + (y - y_0)^2 < \delta^2,$$

that is,

$$|MM_0| < \delta.$$

Definition 1. The number A is called the limit of the function $f(M)$ for M tending to M_0 if for any sequence of points M_n such that $\lim\limits_{n\to\infty} |M_nM_0| = 0$, the following equality is satisfied:

$$\lim_{n\to\infty} f(M_n) = A.$$

In this case we write:

$$\lim_{M\to M_0} f(M) = A.$$

If the point M has the coordinates (x, y), then we write:

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = A.$$

The basic theorems on the limits of functions of one variable are valid for functions of two and more variables as well.

Definition 2. A function $f(M)$ is said to be continuous at a point M_0 if the following equality is satisfied:

$$\lim_{M \to M_0} f(M) = f(M_0).$$

In the case of two variables:

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0).$$

A function is said to be continuous on some set if it is continuous at each point of this set.

Example. Show that the function $f(x, y) = \frac{x}{x^2 - y^2}$ is continuous at any point of its domain of definition.

△ The given function is defined at all points (x, y) except the points at which $x^2 = y^2$. Let (x_0, y_0) be some point from the domain of definition of the function $f(x, y)$, that is, $x_0^2 \neq y_0^2$.

We choose an arbitrary sequence of points $\{(x_n, y_n)\}$ such that $x_n^2 \neq y_n^2$ and $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$, then we have:

$$\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} \frac{x_n}{x_n^2 - y_n^2} = \frac{x_0}{x_0^2 - y_0^2} = f(x_0, y_0).$$

Consequently, the given function is continuous at an arbitrary point (x_0, y_0) from the domain of definition. Note that here we have made use of the properties of limits of numerical sequences. ▲

The same as for functions of one variable, the following assertions are proved for functions of several variables: the sum and product of two functions continuous in some domain G are continuous functions in this domain.

The ratio of two functions continuous in the domain G is a function continuous at all points of the domain G at which the denominator does not vanish.

Exercises

1. Find and depict the domain of definition of each of the given functions:

(a) $u = \sqrt{1 - x^2 - 9y^2}$; (b) $u = \frac{1}{2x^2 + 3y^2}$;

(c) $u = \ln(x + y)$; (d) $u = \sqrt{1 - x^2 - y^2 - z^2}$.

2. Find the following limits:

(a) $\lim\limits_{\substack{x\to 0\\ y\to 1}} (x^2+y^2)(x-3y)$; (b) $\lim\limits_{\substack{x\to \pi\\ y\to \frac{1}{2}}} \sin(xy)$.

3. Find out whether the given function is continuous at the point (0, 0):

$$f(x, y)=\begin{cases} \dfrac{4xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x=0,\ y=0. \end{cases}$$

4. Prove that the set of points (x, y) satisfying the inequality $-x^2-y^2>0$ is open.

10.2. Differential Calculus: Functions of Two Variables

1. Partial Derivatives. Consider the function $u=f(x, y)$ defined in some neighbourhood of the point $M_0(x_0, y_0)$. With a fixed value of the variable y, say, $y=y_0$, the function $u=f(x, y_0)$ is a function of one variable x.

The derivative of the function $f(x, y_0)$ with respect to the variable x at the point (x_0, y_0) is called the partial derivative of the function $f(x, y)$ with respect to x at the point (x_0, y_0):

$$\frac{\partial f(x_0, y_0)}{\partial x}=\lim_{x\to x_0}\frac{f(x, y_0)-f(x_0, y_0)}{x-x_0}.$$

The partial derivative of the function $f(x, y)$ with respect to the variable y at the point (x_0, y_0) is defined in a similar manner:

$$\frac{\partial f(x_0, y_0)}{\partial y}=\lim_{y\to y_0}\frac{f(x_0, y)-f(x_0, y_0)}{y-y_0}.$$

For the partial derivatives with respect to the variables x and y we often use other designation:

$$\frac{\partial f}{\partial x}(x_0, y_0),\quad f'_x(x_0, y_0) \text{ and } \frac{\partial f}{\partial y}(x_0, y_0),\quad f'_y(x_0, y_0).$$

In brief, the definition of partial derivatives can be formulated in the following way: f'_x is the derivative of the function $f(x, y)$ with respect to the variable x with the variable y fixed, and f'_y is the derivative of the function $f(x, y)$ with respect to y with x fixed.

Hence it follows that partial derivatives are found by the ordinal differentiation rules for functions of one variable.

In differentiation, for instance, with respect to the variable x, the variable y should be assumed to be constant.

For example, if $f(x, y) = x^2y^3$, then

$$f'_x = (x^2y^3)'_x = 2xy^3,$$
$$f'_y = (x^2y^3)'_y = 3x^2y^2.$$

Similarly, if $z = x^2y + x \sin xy$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + x \sin xy) = 2xy + \frac{\partial}{\partial x}(x \sin xy)$$
$$= 2xy + \sin xy + xy \cos xy,$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y + x \sin xy) = x^2 + x^2 \cos xy.$$

The partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ are functions of two variables. Consequently, for them we also may consider partial derivatives.

Partial derivatives of the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are called the partial derivatives of the second order of the function $f(x, y)$.

The function $f(x, y)$ has four partial derivatives of the second order:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{x}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

They are respectively denoted by

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y\, \partial x}, \quad \frac{\partial^2 f}{\partial x\, \partial y}, \quad \frac{\partial^2 f}{\partial y^2}.$$

The partial derivatives $\frac{\partial^2 f}{\partial y\, \partial x}$ and $\frac{\partial^2 f}{\partial x\, \partial y}$ are called mixed derivatives of the second order. It is possible to prove that if a function $z = f(x, y)$ has continuous partial derivatives, then these derivatives are equal to each other:

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}.$$

Example. Find the second-order partial derivatives of the function $z = e^{xy}$.

△ We first find the partial derivatives of the first order:

$$\frac{\partial z}{\partial x} = ye^{xy}, \quad \frac{\partial z}{\partial y} = xe^{xy}.$$

We now find the second-order partial derivatives with respect to x and y, respectively:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(ye^{xy}) = y^2e^{xy}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(xe^{xy}) = x^2e^{xy}.$$

For mixed derivatives we have:

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}(ye^{xy}) = e^{xy} + xye^{xy},$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + xye^{xy},$$

and, consequently,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = (1 + xy)\,e^{xy}. \blacktriangle$$

2. Geometrical Meaning of Partial Derivatives. Geometrically, the function $z = f(x, y)$ can be represented as a surface in space in which an xyz-system of rectangular coordinates is introduced.

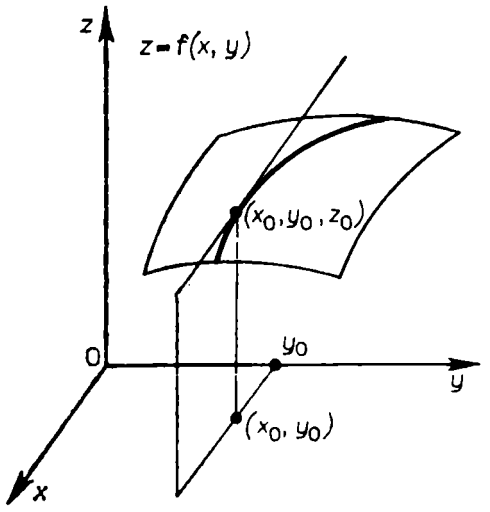

Fig. 134

Let the letter A denote some point of the surface under consideration with coordinates x_0, y_0, and $z_0 = f(x_0, y_0)$.

Through the point A, we draw a plane parallel to the z-axis and perpendicular to the y-axis, whose equation has the form $y = y_0$. The intersection of the plane and the given surface will be a curve passing through the point A (x_0, y_0, z_0) and belonging to the surface. In the plane $y = y_0$ this curve (Fig. 134) has the equation

$$z = f(x, y_0). \qquad (1)$$

The slope of the tangent to the curve defined by equation (1) at the point with coordinates (x_0, y_0, z_0) will be equal to

$$k_{0x} = \frac{\partial f}{\partial x}(x_0, y_0). \qquad (2)$$

Exactly in the same way, we can show that the slope of the tangent to the intersection of the given surface and the plane parallel to the zy-plane at the point (x_0, y_0, z_0) will be equal to

$$k_{0y} = \frac{\partial f}{\partial y}(x_0, y_0).$$

Thus, the first-order partial derivatives of the function $z = f(x, y)$ at the point (x_0, y_0) yield the slopes of the tangents (passing through the point A (x_0, y_0, z_0)) to the lines of intersection of the surface $z = f(x)$, and the planes parallel to the xz- and yz-planes, respectively.

3. Differentiability and the Total Differential. Consider the increment of the function $u = f(x, y)$ at the point $M(x, y)$ corresponding to the increments of the arguments Δx and Δy:

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Definition 1. A function $u = f(x, y)$ is said to be differentiable at a point $M(x, y)$ if its increment at this point can be represented in the form

$$\Delta u = A\Delta x + B\Delta y + \alpha_1(\Delta x, \Delta y)\cdot\Delta x + \alpha_2(\Delta x, \Delta y)\cdot\Delta y, \qquad (1)$$

where A and B are some numbers independent of Δx and Δy, and α_1 and α_2 are infinitesimals as $\Delta x \to 0$ and $\Delta y \to 0$.

The first two terms in expression (1) are called the principal linear part of the increment in the function at the point (x, y).

Definition 2. The total differential of a function of two variables is the principal linear part of its increment.

The differential of the function $u = f(x, y)$ is denoted by the symbol du. Thus, the differential of the function $u = f(x, y)$ is defined by the formula

$$du = A\Delta x + B\Delta y.$$

Let us prove the necessary condition of the differentiability of a function at a point.

Theorem 10.2.1. *If a function $u = f(x, y)$ is differentiable at a point $M(x, y)$, then this function has partial derivatives with respect to x and y at the given point, and the coefficients A and B in the principal linear part of the increment in the function are computed by the formulas*

$$A = \frac{\partial f}{\partial x}(x, y), \quad B = \frac{\partial f}{\partial y}(x, y).$$

□ By hypothesis, the function $u = f(x, y)$ is differentiable at the point (x, y). This means that its increment can be represented in the form (1):

$$\Delta u = A\Delta x + B\Delta y + \alpha_1\Delta x + \alpha_2\Delta y. \qquad (2)$$

Since the increments in the arguments Δx and Δy are arbitrary, we set in equality (2) the increment in the argument y to be equal to zero, that is, $\Delta y = 0$. Then it takes the form

$$\Delta_x u = A\Delta x + \alpha_1\cdot\Delta x. \qquad (3)$$

Here, the symbol $\Delta_x u$ denotes the partial increment in the function u corresponding to the increment in the argument x, equal to Δx, and to the increment in the argument y equal to zero:

$$\Delta_x u = f(x + \Delta x, y) - f(x, y).$$

We divide both sides of equality (3) by Δx and pass to the limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} = \lim_{\Delta x \to 0} (A + \alpha_1). \qquad (4)$$

By hypothesis, α_1 is an infinitesimal function, that is, it tends to zero as Δx tends to zero. The coefficient A is independent of Δx. Computing the limits in equality (4), we get

$$A = \frac{\partial u}{\partial x}(x, y),$$

that is, the coefficient A is equal to the partial derivative of the function $u = f(x, y)$ with respect to the variable x at the point (x, y). Reasoning in a similar way with respect to the independent variable y we get for the coefficient B:

$$B = \frac{\partial f}{\partial y}(x, y).$$

Thereby the theorem has been proved.

As a consequence of the proved theorem, we get a simple formula for computing the differential. To this end, we substitute into the expression for the total differential

$$du = A\Delta x + B\Delta y$$

the expressions of the coefficients A and B in terms of the partial derivatives:

$$du = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y. \qquad (5)$$

The increments in the independent variables Δx and Δy are denoted by the symbols dx and dy.

As a result, we get the expression for the total differential in the form

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \qquad (6)$$

4. The Sufficient Condition for the Differentiability of a Function of Two Variables. Consider the increment Δu in the function $u = f(x, y)$:

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Adding and subtracting $f(x, y + \Delta y)$ on the right-hand side of the given equality, we get the formula

$$\Delta u = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]. \qquad (1)$$

In the first brackets we have the increment in the function u with the variable y fixed, and in the second brackets the increment in the same function with the variable x fixed.

Let the function $f(x, y)$ have continuous partial derivatives in the neighbourhood of the point (x, y). We apply Lagrange's formula to either of the increments, which is possible since in each of the two brackets of expression (1) increments are taken only with respect to one independent variable. Then we get the equality

$$\Delta u = f'_x(x + \theta\Delta x, y + \Delta y)\,\Delta x + f'_y(x, y + \theta_1\Delta y)\,\Delta y, \qquad (2)$$

where θ and θ_1 lie between zero and unity.

From the assumption of the continuity of the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ it follows that, as Δx and Δy tend to zero, the coefficient of Δx will tend to $f'_x(x, y)$, and the coefficient of Δy to $f'_y(x, y)$. Then, by the theorem on the limit of a function of two variables (which is similar to the corresponding theorem for a function of one variable), equality (2) takes the form

$$\Delta u = (f'_x(x, y) + \alpha_1)\,\Delta x + (f'_y(x, y) + \alpha_2)\,\Delta y,$$

or

$$\Delta u = f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y + \alpha_1 \cdot \Delta x + \alpha_2 \cdot \Delta y,$$

where α_1 and α_2 are infinitesimal functions as $\Delta x \to 0$ and $\Delta y \to 0$. Thus, the increment Δu has been represented in form (1) in the preceding subsection.

The first two terms of this formula yield the total differential to the function $u = f(x, y)$:

$$du = f'_x(x, y)\,dx + f'_y(x, y)\,dy.$$

Thus, we have proved the following theorem:

Theorem 10.2.2. *If a function of two variables $u = f(x, y)$ has continuous first-order partial derivatives with respect to the variables x and y in some neighbourhood of a point, then the total differential of the function $u = f(x, y)$ exists at this point and is computed by the formula*

$$du = f'_x(x, y)\,dx + f'_y(x, y)\,dy.$$

This theorem expresses the sufficient test for a function to be differentiable.

Example 1. Find the total differential of the function $u = x \sin y$ at the points (1, 0) and (1, 1).

△ First we find the partial derivatives:

$$\frac{\partial u}{\partial x} = \sin y, \quad \frac{\partial u}{\partial y} = x \cos y.$$

They are continuous for any x and y. Consequently, the total differential

$$du = \sin y \cdot dx + x \cos y \cdot dy$$

exists at any point (x, y). To find the differential of a given point, we have to substitute the coordinates of this point for x and y. Thus,

$$du\ (1,\ 0) = \sin 0 \cdot dx + 1 \cdot \cos 0 \cdot dy = dy;$$

$$du\ (1,\ 1) = \sin 1 \cdot dx + 1 \cdot \cos 1 \cdot dy = \sin 1 \cdot dx + \cos 1 \cdot dy. \ \blacktriangle$$

From formula (1) in the preceding subsection it follows that

$$\Delta u\ (x_0,\ y_0) - du\ (x_0,\ y_0) = \alpha_1 \cdot \Delta x + \alpha_2 \cdot \Delta y, \qquad (3)$$

where $\alpha_1 \to 0$ and $\alpha_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, so that the total increment in the function at the point (x_0, y_0) is approximately equal to the differential of this function at the same point. Hence we find the formula for approximate computations:

$$u\ (x,\ y) \approx u\ (x_0,\ y_0) + \frac{\partial u\ (x_0,\ y_0)}{\partial x} \cdot (x - x_0) + \frac{\partial u\ (x_0,\ y_0)}{\partial y} \cdot (y - y_0), \qquad (4)$$

where we have set $\Delta x = x - x_0$, $\Delta y = y - y_0$ and taken into account that $\Delta u\ (x_0,\ y_0) = u\ (x_0 + \Delta x,\ y_0 + \Delta y) - u\ (x_0,\ y_0) = u\ (x,\ y) - u\ (x_0,\ y_0)$.

Example 2. Evaluate the function $u = \sqrt{x^2 + y^2}$ at the point (2.8, 4.1).

$\triangle$ The given point lies near the point $(x_0, y_0) = (3, 4)$ at which the function is readily evaluated. The same refers to its first partial derivatives $u'_x = \dfrac{x}{\sqrt{x^2 + y^2}}$ and $u'_y = \dfrac{y}{\sqrt{x^2 + y^2}}$:

$$u\ (3,\ 4) = 5; \quad u'_x\ (3,\ 4) = 0.6; \quad u'_y\ (3,\ 4) = 0.8.$$

Making use of formula (4), we find:

$$\sqrt{2.8^2 + 4.1^2} \approx 5 + 0.6 \cdot (2.8 - 3) + 0.8 \cdot (4.1 - 4) = 4.96. \ \blacktriangle$$

5. Differentiation of a Composite Function. Suppose that the function $u = f\ (x, y)$ has continuous first partial derivatives with respect to the variables x and y in some neighbourhood of the point $M\ (x, y)$. Suppose also that the arguments x and y are differentiable functions of the independent variable t, that is, $x = x\ (t)$, $y = y\ (t)$. Then the function $u = f\ (x, y)$ is a composite function of the independent variable t: $u = f\ (x\ (t),\ y\ (t))$. Let us derive the formula for computing the derivative $\dfrac{du}{dt}$. We assign an increment Δt to the independent variable t, then the arguments x and y of the function $u = f\ (x, y)$ will gain increments Δx and Δy, respectively, and the

function $u = f(x, y)$ will gain an increment Δu:

$$\Delta u = f(x + \Delta x, \quad y + \Delta y) - f(x, y).$$

In accordance with formula (2) of the preceding subsection, this increment can be written in the form

$$\Delta u = f'_x(x + \theta\Delta x, \ y + \Delta y)\,\Delta x + f'_y(x, \ y + \theta_1\Delta y)\,\Delta y.$$

We then divide both sides of this equality by Δt:

$$\frac{\Delta u}{\Delta t} = f'_x(x + \theta\Delta x, \ y + \Delta y)\frac{\Delta x}{\Delta t} + f'_y(x, \ y + \theta_1\Delta y)\frac{\Delta y}{\Delta t}. \qquad (1)$$

Since the variables x and y have a derivative with respect to the variable t, they will also be continuous functions of the variable t. Therefore as Δt tends to zero the increments Δx and Δy will also tend to zero.

Passing to the limit as $\Delta t \to 0$ in equality (1) and taking into account the continuity of the first derivatives of the function $u = f(x, y)$ at the point (x, y), we get the desired formula:

$$\frac{du}{dt} = f'_x(x, y)\frac{dx}{dt} + f'_y(x, y)\frac{dy}{dt}. \qquad (2)$$

Formula (2) is often written in the form

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial y}{\partial y}\cdot\frac{du}{dt}. \qquad (3)$$

Example 1. Find the derivative $\frac{du}{dt}$ of the function

$$u = e^{x+2\sin y},$$

if $x = e^t$, and $y = \ln t$.

△ For computing the derivative $\frac{du}{dt}$, it is convenient to make use of formula (3):

$$\frac{\partial u}{\partial x} = e^{x+2\sin y}; \quad \frac{\partial u}{\partial y} = e^{x+2\sin y}\cdot 2\cos y;$$

$$\frac{dx}{dt} = e^t; \quad \frac{dy}{dt} = \frac{1}{t}$$

and finally

$$\frac{du}{dt} = e^{t+x+2\sin y} + \frac{2\cos y}{t}\cdot e^{x+2\sin y} = \left(e^t + \frac{2\cos y}{t}\right)e^{x+2\sin y}. \ \blacktriangle$$

If the part of the independent variable t is played by the variable x, then the function $u = f(x, y)$, where $y = y(x)$, is a composite function of x. In this case, formula (2) will take the form:

$$\frac{du}{dx} = f'_x + f'_y\frac{dy}{dx},$$

since $\frac{dx}{dx} = 1$. Let us now consider a more general case when the variables x and y are functions of two other independent variables:

$$x = x(s, t), \quad y = y(s, t).$$

Let us derive the formulas for computing the derivatives $\frac{du}{ds}$ and $\frac{\partial u}{\partial t}$. When computing $\frac{\partial u}{\partial s}$, we assume t to be constant, and when differentiating with respect to the variable t, s is assumed to be constant. Therefore, when computing the derivatives $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ we may apply formula (2):

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

These formulas are often written as follows:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}, \tag{4}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Example 2. Find the first partial derivatives of the composite function $u = e^{x+y}$ with respect to the variables s and t if

$$x = t + \sin s, \qquad y = t + \cos s.$$

△ Applying formulas (4) to the given function, we get the expressions for the partial derivatives:

$$\frac{\partial u}{\partial s} = e^{x+y} \cdot \cos s + e^{x+y}(-\sin s),$$

$$\frac{\partial u}{\partial t} = e^{x+y} \cdot 1 + e^{x+y} \cdot 1 = 2e^{x+y}. \ ▲$$

Exercises

1. Find the partial derivatives of the given functions and their total differentials:

(a) $u = x^3 - y^3 + 4xy$; (b) $u = \sqrt{x + 2xy + z}$,

(c) $u = \arctan\left(\frac{2y}{x}\right)$; (d) $u = x^2y + \frac{x}{y}$.

2. Find the partial derivatives of the following functions with respect to the variables s and t:

(a) $u = \sqrt{x + y}$, where $x = \cos(s - t)$, $y = \sin(s + 2t)$;

(b) $u = x^2y$, where $x = e^{s+3t}$, $y = \ln(t + 6s)$.

3. Find the partial derivatives of the second order of the given functions: (a) $u = \sqrt{x^2+y^2+z^2}$; (b) $u = e^{x^2-y}$; (c) $u = \ln(x + \sqrt{x-y})$.

10.3. Local Extremum of a Function of Two Variables. Taylor's Formula

1. Extrema of a Function of Two Variables. Let us give the definition of points of minimum and maximum of a function of two variables $u = f(x, y)$.

Definition. A point $M_0(x_0, y_0)$ from the domain of definition of a function $f(x, y)$ is called a point of minimum (maximum) of this function if there is a δ-neighbourhood of this point such that for all points $M(x, y) \neq M_0(x_0, y_0)$ from this δ-neighbourhood the inequality

$$f(x, y) > f(x_0, y_0) \quad (f(x, y) < f(x_0, y_0))$$

is satisfied.

The points of minimum and maximum of a function of two variables are called the points of extremum of this function, and the values of the function at these points are respectively called the minimum and maximum or extrema of the function.

The same as in the case of a function of one variable, it may happen that a certain maximum of a function of two variables will turn out to be less than some minimum. However, there is no contradiction here, since an extremum is connected with a certain neighbourhood of the given point from the domain of definition of the function, but not with the entire domain. Therefore, to designate this notion, the term "local extremum" is used.

Now, let us consider the necessary condition for existence of an extremum of a function of two variables. Let the function $f(x, y)$ of two variables have a maximum or a minimum at the point (a, b), that is, let the point (a, b) be a point of local extremum. We then fix one of the arguments of the function $u = f(x, y)$, for instance, we set the variable y to be equal to the constant b. In this case, the function $f(x, b)$ will be a function of one variable x. This function $f(x, b)$ has a maximum or a minimum at the point $x = a$. Therefore its derivative must vanish or not exist for $x = a$.

Reasoning in a similar way, we make sure that the derivative of the function $f(a, y)$ with respect to the variable y must vanish or not exist for $y = b$.

In the final analysis, we arrive at the necessary condition for an extremum to exist: the function $f(x, y)$ has an extremum only at the points where the partial first-order derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ vanish or do not exist. The points at which the first partial derivatives vanish are called stationary points.

Let us finally formulate the rule for finding the points at which a function can have an extremum: we equate the first partial derivatives to zero and solve the system thus obtained. Its solutions and also the points at which the derivatives do not exist are those values of the independent variables for which the function can attain an extremum.

Example. Find the stationary points of the function

$$u = x^2 + 2xy - y^2 + x.$$

△ We find the first partial derivatives of the function $u = u(x, y)$ and equate them to zero:

$$\begin{cases} 2x + 2y + 1 = 0, \\ 2x - 2y = 0. \end{cases}$$

Solving this system of equations, we get the only stationary point $\left(-\frac{1}{4}, -\frac{1}{4}\right)$. ▲

2. Taylor's Formula for a Function of Two Independent Variables. Taylor's formula yields the expansion of the value $f(x + h, y + k)$ of the function $f(x, y)$ in powers of the increments h and k in the independent variables.

To this end, we consider an auxiliary function of the variable $tF(t) = f(x + ht, y + kt)$ which turns into the given function $f(x + h, y + k)$ for $t = 1$.

We apply to the function $F(t)$ Maclaurin's formula

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \frac{F''(\theta t)}{2!}t^2, \quad 0 < \theta < 1. \tag{1}$$

where θt lies between zero and t (see 6.11, subsection 1).

Using the formula for differentiating composite functions of two variables (see 10.2, subsection 5), we find:

$$F'(t) = h\frac{\partial f}{\partial x}(x + ht, y + kt) + k\frac{\partial f}{\partial y}(x + ht, y + kt),$$

so that $F'(0) = h\frac{\partial f}{\partial x}(x, y) + k\frac{\partial f}{\partial y}(x, y)$.

We now compute the second derivative of the function $F(t)$:

$$F''(t) = h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}.$$

Substituting the derivatives F'_t and F''_t into formula (1) and setting then $t = 1$, we get Taylor's formula:

$$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x}(x, y) + k\frac{\partial f}{\partial y}(x, y) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x+\theta h, y+\theta k)\cdot h^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}(x+\theta h, y+\theta k)\,hk + \frac{\partial^2 f}{\partial y^2}(x+\theta h, y+\theta k)\cdot k^2\right).$$

3. The Sufficient Condition for a Function of Two Variables to Have an Extremum. Suppose that the function $u = f(x, y)$ has continuous second partial derivatives in the neighbourhood of the stationary point (x_0, y_0).

Taylor's formula for the function $u = f(x, y)$ in the neighbourhood of this point (taking into account that the first derivatives at it are equal to zero) will take the form:

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x_0+\theta h, y_0+\theta k) + hk\frac{\partial^2 f}{\partial x\,\partial y}(x_0+\theta h, y_0+\theta k) + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2}(x_0+\theta h, y_0+\theta k).$$

In the case of a minimum for any sufficiently small values of h and k the right-hand side must be positive, and in the case of a maximum, negative.

By virtue of supposition on the continuity of the second partial derivatives, it suffices that the expression

$$R = h^2\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + 2hk\frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0) + k^2\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

retains the same sign for small (in absolute value) values of h and k.

Let us analyze the conditions imposed by this requirement on the derivatives on the right-hand side of this equality. Let us assume that one of the derivatives computed at the stationary point:

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0),$$

for instance A, does not vanish.

Denoting the value of the mixed derivative $\frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0)$ by B, we can bring the expression for R to the form

$$R = A\left(h + k\frac{B}{A}\right)^2 + k^2\left(\frac{AC-B^2}{A}\right).$$

Hence we find that the sufficient condition for a maximum to exist will be the fulfillment of the inequalities

$$AC - B^2 > 0, \quad A < 0.$$

Similarly, the sufficient condition for a minimum to exist will be the fulfillment of the inequalities

$$AC - B^2 > 0, \quad A > 0.$$

We denote the expression $AC - B^2$ by D. Then

$$D = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0)\right)^2,$$

or in the form of a determinant

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial x\,\partial y}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{vmatrix}.$$

Thus, we have the following rule for analyzing differentiable functions of two variables for an extremum at critical points.

Compute the determinant

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix}.$$

If it is positive, then the function $z = f(x, y)$ has an extremum at the point (x_0, y_0). If the derivative A is negative, the function attains a maximum at the point (x_0, y_0), and if A is positive, then the function has a minimum at this point.

If the determinant D is negative, then the function $z = f(x, y)$ has no extremum at the point (x_0, y_0).

Finally, if the determinant D is equal to zero, then a stationary point is tested for an extremum with the help of derivatives of higher orders.

Example 1. Test the function $u = x^2 + 2y^2 + x$ for an extremum.

△ We find the first-order partial derivatives and equate them to zero:

$$\begin{cases} 2x + 1 = 0, \\ 4y = 0. \end{cases}$$

The stationary point of the given function has the coordinates $\left(-\frac{1}{2}, 0\right)$. We then compute the second derivatives at the

stationary point:

$$A=\frac{\partial^2 f\left(-\frac{1}{2},\ 0\right)}{\partial x^2}=2,\quad B=\frac{\partial^2 f\left(-\frac{1}{2},\ 0\right)}{\partial x\,\partial y}=0,$$

$$C=\frac{\partial^2 f\left(-\frac{1}{2},\ 0\right)}{\partial y^2}=4,$$

and the determinant:

$$D=\begin{vmatrix}2 & 0\\ 0 & 4\end{vmatrix}=8.$$

Since the determinant $D>0$, the given function has an extremum at the point $\left(-\frac{1}{2}, 0\right)$. Since $A>0$, a minimum occurs at this point. ▲

Example 2. Test the function $z = 4xy + x^2$ for an extremum.
△ We find the stationary points of the given function:

$$\begin{cases} 4y+2x=0,\\ 4x=0, \end{cases}$$

whence $x = 0$ and $y = 0$. We compute the second derivatives at the stationary point (0, 0):

$$A = 2,\quad B = 4,\quad C = 0,$$

and the determinant:

$$D=\begin{vmatrix}2 & 4\\ 4 & 0\end{vmatrix}=-16.$$

We see that the determinant D is negative, and this means that the given function has no extremum at the point (0, 0). ▲

Example 3. Test the function $u = (x + y^2)e^x$ for an extremum.
△ We find the first-order partial derivatives and equate them to zero:

$$\begin{cases} (1+x+y^2)\,e^x=0,\\ 2ye^x=0, \end{cases}$$

whence we find the only stationary point (— 1, 0).
We then find the second-order derivatives:

$$\frac{\partial^2 u}{\partial x^2}=(2+x+y^2)\,e^x,\quad \frac{\partial^2 u}{\partial x\,\partial y}=2ye^x,\quad \frac{\partial^2 u}{\partial y^2}=2e^x.$$

We now compute the values of these derivatives at the stationary point:

$$A = \frac{\partial^2 u(-1, 0)}{\partial x^2} = e^{-1}, \quad B = \frac{\partial^2 u(-1, 0)}{\partial x \partial y} = 0,$$

$$C = \frac{\partial^2 u(-1, 0)}{\partial y^2} = 2e^{-1}.$$

Since $D = AC - B^2 = \frac{2}{e^2} > 0$ and $A = \frac{1}{e} > 0$, at the point $(-1, 0)$ the function $u = (x + y^2)\, e^x$ attains a minimum equal to $u(-1, 0) = (-1 + 0)\, e^{-1} = -\frac{1}{e}$. ▲

Exercises

1. Test the following functions for extrema:

(a) $u = x^2 + y^2 + 3x - 4y$; (b) $u = x^2 - y^2 + 3x - 4y$; (c) $u = x^4 + 4xy - 2y^2$; (d) $u = x^2 + y^2 - xy + x$.

2. Show that the function $z = xy$ has no extremum at the origin. Sketch the graph of the given function.

3. Show that the function $u = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$ has a minimum at the origin.

4. Find the points of extremum of the function $u = \sqrt{x^2 + y^2}$.

10.4. The Equation of a Tangent Line to a Curve in Space. The Equations of a Tangent Plane and Normal to a Surface

1. The Equation of a Tangent Line to a Curve in Space. A curve in space can be represented parametrically by the equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where t is some parameter. First, as a parameter, we take the arc length s counted from some fixed point M_0 to the point M.

The equations of the curve in space will take the following form:

$$x = x(s), \quad y = y(s), \quad z = z(s).$$

Let us find out the geometrical meaning of the derivatives of x, y, and z with respect to the variable s.

Consider two neighbouring points N and M on the curve situated in the order of increase in the parameter s.

As N tends to M, the direction of the vector $\overrightarrow{NM}$ in the limit specifies the direction of the tangent line. This direction will be called the positive direction of the tangent.

Let α_1 denote the angle formed by the vector $\overrightarrow{NM}$ with the positive direction of the x-axis, and let Δx denote the projection of the

vector $\overrightarrow{NM}$ on the x-axis. Then

$$\Delta x = |\overrightarrow{NM}| \cdot \cos \alpha_1. \tag{1}$$

The length of the vector $|\overrightarrow{NM}|$ is computed in terms of its projections on the coordinate axes by the formula

$$|\overrightarrow{NM}| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

We divide both sides of equality (1) by the length of the arc Δs connecting the points N and M:

$$\frac{\Delta x}{\Delta s} = \frac{|\overrightarrow{NM}|}{\Delta s} \cos \alpha_1. \tag{2}$$

Let us pass to the limit in equality (2) as $\Delta s \to 0$. We take without proof that the ratio of the arc length NM, equal to Δs, to the length of the vector $\overrightarrow{NM}$ tends to unity. The angle α_1 tends to the angle α formed by the positive direction of the tangent with the positive direction of the x-axis. Thus, we get

$$\frac{dx}{\partial s} = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \cos \alpha.$$

Similarly, we can show that

$$\frac{dy}{ds} = \cos \beta, \quad \frac{dz}{ds} = \cos \gamma,$$

where β and γ are angles between the tangent and the y- and z-axes, respectively.

The direction vector of the tangent line to the given curve at the point (x_0, y_0, z_0) will be represented by the vector

$$\mathbf{t} = \{\cos \alpha, \cos \beta, \cos \gamma\} = \{x'_s, y'_s, z'_s\}. \tag{3}$$

Then the canonical equations of the tangent line take the form (see 3.2):

$$\frac{x - x_0}{x'_s} = \frac{y - y_0}{y'_s} = \frac{z - z_0}{z'_s}. \tag{4}$$

If now, instead of the parameter s characterizing the position of a variable point M on the given curve, we introduce another parameter t, then this parameter t will be a function of the parameter s, that is, $t = t(s)$.

The derivatives x'_s, y'_s, z'_s are computed by the rule for differentiating a composite function:

$$x'_s = x'_t \cdot t'_s, \quad y'_s = y'_t \cdot t'_s, \quad z'_s = z'_t \cdot t'_s \tag{5}$$

Substituting derivatives (5) into equations (4) and reducing by the common factor t'_s, we get the equations of a tangent line to a curve (in the general case):

$$\frac{x-x_0}{x'_t} = \frac{y-y_0}{y'_t} = \frac{z-z_0}{z'_t}.$$

2. The Equations of a Tangent Plane and a Normal to a Surface. Let us introduce the notion of a tangent plane to the surface given by the equation

$$F(x, y, z) = 0. \tag{1}$$

Let $M(x_0, y_0, z_0)$ denote some point of the surface, and L the curve drawn on the surface through the point M.

The curve L in parametric form is defined by the equations

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

Since the curve L lies on the given surface, the coordinates of its points $x(t)$, $y(t)$, and $z(t)$ satisfy equation (1):

$$F(x(t), \quad y(t), \quad z(t)) = 0.$$

Differentiating both sides of this equality, on the basis of the formula for computing the derivative of a composite function, we get:

$$\frac{dF}{dt} = F'_x x'_t + F'_y y'_t + F'_z z'_t = 0. \tag{2}$$

Suppose that all the derivatives in equality (2) are continuous, and also

$$(x'_t)^2 + (y'_t)^2 + (z'_t)^2 \neq 0 \quad \text{and} \quad (F'_x)^2 + (F'_y)^2 + (F'_z)^2 \neq 0.$$

Equality (2) is the condition of the orthogonality of the vectors $\{F'_x(M), F'_y(M), F'_z(M)\}$ and $\{x'_t, y'_t, z'_t\}$.

As is shown in the preceding subsection, the vector $\{x'_t, y'_t, z'_t\}$ is the direction vector of the tangent line to the curve L.

Equation (2) shows that the tangent lines to all possible curves passing through the point M are orthogonal to the same vector $\{F'_x(M), F'_y(M), F'_z(M)\}$. Then they lie in the same plane which is called the tangent plane to the surface at the point M.

The vector $\{F'_x(M), F'_y(M), F'_z(M)\}$ will be the vector orthogonal to the tangent plane.

Thus, the equation of the tangent plane to the surface at the point M will have the form (see 3.1, subsection 2):

$$F'_x(M)(x - x_0) + F'_y(M)(y - y_0) + F'_z(M)(z - z_0) = 0.$$

The normal to a surface is defined as a straight line perpendicular to the tangent plane. Consequently, the vector $\{F'_x(M), F'_y(M), F'_z(M)\}$ will be the direction vector of the normal.

The normal passes through the point M. Thus, its canonical equations have the form:

$$\frac{x-x_0}{F'_x(M)} = \frac{y-y_0}{F'_y(M)} = \frac{z-z_0}{F'_z(M)}.$$

Suppose that a surface is given by an equation in explicit form: $z = f(x, y)$. The implicit equation of the surface in this case will be written in the form

$$F(x, y, z) = f(x, y) - z = 0.$$

The partial derivatives of the function F at the point M will be equal to $F'_x(x_0, y_0, z_0) = f'_x(x_0, y_0)$, $F'_y(x_0, y_0, z_0) = f'_y(x_0, y_0)$, $F'_z(x_0, y_0, z_0) = -1$.

Let p and q denote the partial derivatives $f'_x(x_0, y_0)$ and $f'_y(x_0, y_0)$, respectively. Then the equation of the tangent plane takes the form:

$$p(x - x_0) + q(y - y_0) - (z - z_0) = 0,$$

since $\mathbf{n} = \{p, q, -1\}$ is the normal vector of this plane.

The equations of the normal to the surface will be written in the form:

$$\frac{x-x_0}{p} = \frac{y-y_0}{q} = \frac{z-z_0}{-1}.$$

The vector $\mathbf{n} = \{p, q, -1\}$ is the direction vector of this straight line.

The direction cosines of the normal to the surface, that is, the cosines of the angles made by this straight line with the coordinate axes, will be equal to

$$\cos\widehat{(\mathbf{n}, \mathbf{i})} = \frac{p}{\sqrt{1+p^2+q^2}}, \quad \cos\widehat{(\mathbf{n}, \mathbf{j})} = \frac{q}{\sqrt{1+p^2+q^2}},$$

$$\cos\widehat{(\mathbf{n}, \mathbf{k})} = \frac{-1}{\sqrt{1+p^2+q^2}},$$

since $\mathbf{ni} = |\mathbf{n}| \cos\widehat{(\mathbf{n}, \mathbf{i})}$, $\mathbf{nj} = |\mathbf{n}| \cos\widehat{(\mathbf{n}, \mathbf{j})}$, $\mathbf{nk} = |\mathbf{n}| \cos\widehat{(\mathbf{n}, \mathbf{k})}$, and $|\mathbf{n}| = \sqrt{1+p^2+q^2}$.

Exercises

1. Write the equation of the tangent line to the helical line $x = a\cos\varphi$, $y = a\sin\varphi$, $z = h\varphi$ at the point corresponding to $\varphi = \frac{\pi}{4}$. Sketch the helical line.

2. Write the equation of the tangent plane and the equation of the normal to the given surface at the indicated point:

(a) to the paraboloid $u = 4x^2 + 6y^2$ at the point (1, 1, 10);
(b) to the surface $u = xy + 3$ at the point (2, 2, 7).
Sketch the surface and the tangent plane at the indicated point.

3. The surfaces $x^2 + y^2 + z^2 = 3$, $xyz = 1$, and $z = xy$ pass through the point (1, 1, 1). Find the angles between them.

Chapter 11

DIFFERENTIAL EQUATIONS

11.1. Differential Equations and Their Properties

1. Problems Leading to Differential Equations. Many problems of natural science are formulated as equations containing both the unknown quantities (desired functions) and the rates of change of these quantities (that is, the derivatives of the desired functions).

Definition 1. Equations in which the unknown function appears under the sign of the derivative or the differential are called differential equations.

Thus, many problems of natural science are formulated as differential equations. If an equation contains an independent variable, the unknown function and its first derivative (but no higher-order derivatives), then this equation is called the differential equation of first order. If, in addition, an equation contains the derivative of the second order of the unknown function, then the equation is called the differential equation of second order. Differential equations of third, fourth, etc. order are defined in a similar manner.

Definition 2. The order of a differential equation is the highest order of the derivative (or differential) of the unknown function.

Let us consider some examples.

Let a body of mass m fall from a certain height. It is required to find the law of change of free-fall velocity v with time t, that is, we have to find the function $v = v(t)$. The fundamental law of mechanics is represented by Newton's second law:

$$m \cdot \frac{dv}{dt} = F, \tag{1}$$

where $\frac{dv}{dt} = a$ is the free-fall acceleration of the body (the derivative of the velocity v with respect to time t), and F the resultant force acting on the body during the process of motion. In our case

$$F = mg - F_{\text{res}}, \tag{2}$$

where mg is the force of gravity, and F_{res} the resistance force of air. As is known, in the case of a streamlined body and moderate velocities of motion the resistance force of air is proportional to the velocity of the moving body, that is,

$$F_{\text{res}} = \rho v, \tag{3}$$

where ρ is a proportionality factor. Substituting formulas (2) and (3) into Newton's equation (1), we find the first-order differential

equation describing the free fall of the body in air:

$$\frac{dv}{dt} = g - \frac{\rho}{m} v. \tag{4}$$

If we solve differential equation (4), that is, if we find the function $v = v(t)$ turning this equation into an identity with respect to the independent variable t on some interval (a, b), then we shall find the desired law of change in the velocity of falling body.

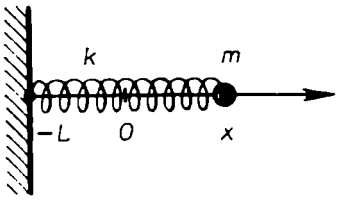

Fig. 135

Let us now consider rectilinear oscillatory motion of a load of mass m fixed on a spring with stiffness k (Fig. 135). The position of the load on the straight line is characterized by its coordinate x varying with time t, that is, by the function $x = x(t)$. As the origin, we take the point O coinciding with the position of the load when the spring is not stretched. If the length of nonstretched spring is L, then the coordinate of the point at which the second end of the spring is fixed is equal to $-L$. Thus, the coordinate of the load x will be numerically equal to the change in length of the spring. Then, by the Hooke law, for moderate stretchings (compressions) the force F exerted by the spring on the load will be

$$F = -kx.$$

Here the minus sign means that the direction of the force is opposite to the direction of stretching (compressing) the spring.

Since, by definition,

$$v = \frac{dx}{dt},$$

Newton's equation (1) in this case takes the form of the second-order differential equation

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{5}$$

2. The General Solution of a Differential Equation. In the general case, a first-order differential equation can be written in the form

$$F(t, x, x') = 0, \tag{1}$$

where $x = x(t)$ is the desired unknown function, $x' = x'(t) = \frac{dx}{dt}$ its derivative with respect to the independent variable t, and F the given function of three variables t, x, and x'. Differential equation (1) is usually reduced to the form

$$x' = f(t, x). \tag{2}$$

Such equations are said to be solved with respect to the derivative. Henceforward, we shall consider only this type of differential equations.

Definition 1. The function $x = \varphi(t)$, $t \in (a, b)$, is called a solution of differential equation (2) if it has a derivative $\varphi'(t)$ on the interval (a, b) and if for any $t \in (a, b)$ the following equality is valid:

$$\varphi'(t) = f(t, \varphi(t)).$$

When the function f in equation (2) depends only on the variable t, we get the simplest first-order differential equation

$$x'(t) = f(t), \tag{3}$$

where $f(t)$ is a given function, and $x(t)$ the desired function. It is easily seen that the problem of finding the solutions of this equation is reduced to finding the antiderivatives of the given function $f(t)$, that is, the problem of evaluating the indefinite integral $\int f(t)\,dt$. Thus, the solution of the simplest differential equation (3) has the form:

$$x(t) = \int f(t)\,dt. \tag{4}$$

This formula contains implicitly an arbitrary constant C. Indeed, if $F(t)$ is an antiderivative of the function $f(t)$, then $\int f(t)\,dt = F(t) + C$, and therefore

$$x(t) = F(t) + C. \tag{5}$$

Thus, the simplest first-order differential equation (3) has infinitely many solutions. For the general case, we give the following definition.

Definition 2. The function $x = \varphi(t, C)$, which for each fixed value of C, as a function of t, is a solution of equation (2), is called the general solution of equation (2).

Example. Find the general solution of the differential equation

$$x' = 5\sin t - \frac{2}{1+t^2}.$$

△ The general solution $x(t)$ of the given differential equation is written in the form of the indefinite integral of the functions on the right-hand side of this equation:

$$x = \int \left(5\sin t - \frac{2}{1+t^2}\right) dt = 5\int \sin t\,dt - 2\int \frac{dt}{1+t^2}$$
$$= -5\cos t - 2\arctan t + C.$$

Thus, $x = -5 \cos t - 2 \arctan t + C$, $t \in R$, where C is an arbitrary constant.▲

3. The Initial Conditions and Cauchy's Problem. Every solution of a differential equation obtained from the general solution of a concrete value of the constant of integration C is called a partial solution.

In most cases, to single out a unique solution from the general solution of a differential equation, we use an additional condition

$$x(t_0) = x_0, \tag{1}$$

where t_0 and x_0 are given numbers. Condition (1) is called the initial condition.

Definition. The problem of finding the solution of the differential equation

$$x' = f(t, x),$$

satisfying initial condition (1), is called Cauchy's problem.

The following theorem holds (we give it without proof).

Theorem 11.1.1. *If the functions $f(t, x)$ and $\frac{\partial f(t, x)}{\partial x}$ are continuous in the neighbourhood of the point $M_0(t_0, x_0)$, then Cauchy's problem*

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a unique solution in some neighbourhood of the point $M_0(t_0, x_0)$.

Example. Find the solution to Cauchy's problem

$$x' = \frac{3}{\cos^2 t}, \quad x(0) = 3.$$

△ We first find the general solution of the differential equation containing the constant of integration C:

$$x(t) = \int \frac{3dt}{\cos^2 t} = 3 \tan t + C. \tag{2}$$

Now we find the unique value of the constant of integration C which satisfies the initial condition. For this purpose, we set $t = 0$ in equality (2) and take into consideration the initial condition

$$3 = 3 \cdot \tan(0) + C.$$

Since $\tan(0) = 0$, we have $C = 3$. The solution to the Cauchy problem is: $x(t) = 3(\tan t + 1)$. ▲

Exercises

1. Find the general solution of each of the given differential equations:

(a) $x'(t)=\frac{t-2}{t^3}$; (b) $x'(t)=\cos^2\left(\frac{t}{2}\right)$; (c) $x'(t)=\frac{1}{\sqrt{3-2t}}$;

(d) $x'(t)=\frac{1}{1-5t}$; (e) $x'(t)=\cot t$; (f) $x'(t)=\tan(4t)$;

(g) $x'(t)=\frac{\sin t}{\cos^3 t}$; (h) $x'(t)=te^{-t^2}$; (i) $x'(t)=\frac{e^{\sqrt{t}}}{\sqrt{t}}$;

(j) $x'(t)=\frac{t}{\sqrt{1-t^2}}$.

2. Find the solution to each of the given Cauchy's problems:

(a) $x'(t)=\sin 5t$, $x(\pi)=1$; (b) $x'(t)=\frac{1}{\sin^2 3t}$, $x\left(\frac{\pi}{12}\right)=0$;

(c) $x'(t)=2\sqrt{t}+\frac{7}{t^4}$, $x(1)=2$; (d) $x'(t)=\frac{2}{3+4t}$, $x(0)=0$.

11.2. The Methods of Solving Some First-order Differential Equations

1. The Equation with Variables Separable. If $f(t, x)=f_1(t)\,f_2(x)$, then the first-order differential equation

$$x'(t)=f(t, x)$$

takes the following form:

$$x'(t)=f_1(t)\,f_2(x). \tag{1}$$

Differential equation (1) is called the equation with variables separable.

We divide equation (1) by $f_2(x)$:

$$\frac{x'(t)}{f_2(x)}=f_1(t). \tag{2}$$

Since the variable x is a function (desired) of t, both sides of equation (2) contain equal functions of the same argument. Hence, their indefinite integrals coincide:

$$\int \frac{x'(t)\,dt}{f_2(x(t))}=\int f_1(t)\,dt. \tag{3}$$

Changing the variable of integration (see 7.2, subsection 1) in the integral on the left-hand side of equality (3), we find finally:

$$\int \frac{dx}{f_2(x)}=\int f_1(t)\,dt. \tag{4}$$

The variable x is contained only on the left-hand side of equality (4), while the variable t only on the right-hand side of this equality. Therefore equation (1) is said to be solved by the method of separation of variables.

Formula (4) contains implicitly an arbitrary constant C (the constant of integration). Thus, formula (4) gives the general solution of the differential equation with variables separable (1). Formula (4) is derived in the supposition that $f_2(x) \neq 0$. And if there is a value $x = x_0$ for which $f_2(x_0) = 0$, then, apart from solution (4), differential equation (1) will have one more solution:

$$x(t) = x_0. \tag{5}$$

Since $x'(t) = \frac{dx}{dt}$, the finding of the general solution (4) for the equation with variables separable (1) can be formulated as the following rule according to which we have:

(a) to separate the variables x and t in the equation

$$\frac{dx}{dt} = f_1(t) f_2(x)$$

by dividing this equation by $f_2(x)$ and multiplying it by dt:

$$\frac{1}{f_2(x)} \cdot dx = f_1(t) \cdot dt;$$

(b) to integrate the left-hand and right-hand sides of the obtained equality with respect to x and t, respectively, that is, to find some antiderivative $F_2(x)$ of the function $\frac{1}{f_2(x)}$ and some antiderivative $F_1(t)$ of the function $f_1(t)$, and to write the equation

$$F_2(x) = F_1(t) + C, \tag{6}$$

where C is an arbitrary constant;

(c) to solve equation (6) with respect to x (if it is possible) and to obtain the general solution in the form

$$x = \varphi(t, C).$$

If it is impossible to be done, then equation (6) is said to specify the general solution implicitly.

Example 1. Solve the equation

$$x' = xt^2. \tag{7}$$

△ This is an equation with variables separable. Separating the variables

$$\frac{dx}{x} = t^2\, dt$$

and integrating, we get:

$$\ln |x| = \frac{1}{3} t^3 + C_1,$$

where C_1 is an arbitrary constant. Hence it follows that

$$|x| = e^{\frac{1}{3}t^3 + C_1} = e^{C_1} \cdot e^{\frac{1}{3}t^3} \quad \text{or} \quad x = (\pm e^{C_1}) \cdot e^{\frac{1}{3}t^3}.$$

Denoting $\pm e^{C_1} = C$ (the constant C can take on all real values, that is, $C \in R$), we get the general solution of equation (7):

$$x = Ce^{\frac{t^3}{3}}. \tag{8}$$

The right-hand side of equation (7) vanishes for $x = 0$. Therefore $x(t) = 0$ is a solution of the given differential equation. This solution is obtained from formula (8) for $C = 0$, that is, formula (8) gives all the solutions of equation (7). ▲

The independent variable in differential equations is denoted by x, and the desired function by y.

Example 2. Find all the solutions of the differential equation

$$y' = xy^2.$$

△ Obviously, $y = 0$ is a solution of the given equation. Let now $y \neq 0$. Then

$$\frac{dy}{y^2} = x\,dx,$$

and, consequently,

$$-\frac{1}{y} = \frac{1}{2} x^2 + C_1.$$

Thus, the general solution of the given equation has the form:

$$y = -\frac{2}{x^2 + C},$$

where C is an arbitrary constant. Note that the solution $y = 0$ is obtained from the general solution for no value of the constant C. ▲

Example 3. Solve the differential equation

$$y' = -\frac{x}{y}.$$

△ Separating the variables:

$$y\,dy = -x\,dx$$

and integrating, we get:

$$y^2 + x^2 = C_1.$$

It is obvious that here $C_1 > 0$. Setting $C_1 = C^2$, we find that

$$y = \pm \sqrt{C^2 - x^2}, \quad x \in (-C, C).$$

The given differential equation has no other solutions. ▲

Example 4. Solve the equation

$$y' = \frac{xy \sin x}{1+y}.$$

△ The constant function $y = 0$ is obviously a solution. Let now $y \neq 0$. We separate the variables:

$$\left(1 + \frac{1}{y}\right) dy = x \sin x \, dx.$$

Integrating the left-hand side of this equation with respect to y:

$$\int \left(1 + \frac{1}{y}\right) dy = \int dy + \int \frac{dy}{y} = y + \ln |y| + C_1,$$

and the right-hand side with respect to x:

$$\int x \sin x \, dx = -\int x \, d\cos x = -x \cos x + \int \cos x \, dx$$
$$= -x \cos x + \sin x + C_2,$$

we get the equation

$$y + \ln |y| = \sin x - x \cos x + C, \tag{9}$$

where $C = (C_2 - C_1)$ is an arbitrary constant. To find the general solution of the given differential equation in the form $y = \varphi(x, C)$, we have to solve equation (9) with respect to y. Unfortunately, this is impossible to be done, since the solution cannot be expressed in terms of elementary functions. However, the problem of finding the solution of the differential equation has been already reduced to solving an equation containing no variables. Formula (9) yields the general solution implicitly. ▲

Example 5. Solve the equation

$$x' = x \sin t^2.$$

△ We separate the variables and integrate:

$$\ln |x| = \int \sin t^2 \, dt + C_1.$$

Hence

$$x(t) = Ce^{\int \sin t^2 \, dt},$$

where $C = \pm e^{C_1}$ is an arbitrary constant. As is known from 7.3, the integral $\int \sin t^2 dt$ is "inexpressible", that is, it cannot be expressed in terms of elementary functions. ▲

2. Homogeneous Equations. A function $f(t, x)$ is said to be homogeneous if for any number μ the following identity is valid:

$$f(\mu t, \mu x) = \mu^{\alpha} \cdot f(t, x),$$

where α is some number called the degree of the homogeneous function.

Example 1. Show that t functions

$$f_1(t, x) = \sqrt{t^3 + x^3}, \quad f_2(t, x) = tx - x^2,$$
$$f_3(t, x) = \frac{t^2 + x^2}{tx}$$

are homogeneous. Find their degrees.

△ For the given functions we have:

$$f_1(\mu t, \mu x) = \sqrt[3]{(\mu t)^3 + (\mu x)^3} = \mu \sqrt[3]{t^3 + x^3} = \mu \cdot f_1(t, x),$$
$$f_2(\mu t, \mu x) = (\mu t) \cdot (\mu x) - (\mu x)^2 = \mu^2 \cdot f_2(t, x),$$
$$f_3(\mu t, \mu x) = \frac{(\mu t^2) + (\mu x)^2}{(\mu t) \cdot (\mu x)} = \frac{t^2 + x^2}{tx} = f_3(t, x).$$

Thus, all the given functions are homogeneous: the first with $\alpha = 1$, the second with $\alpha = 2$, and the third with $\alpha = 0$. ▲

Definition. The differential equation

$$x' = f(t, x) \tag{1}$$

is said to be homogeneous if the function $f(t, x)$ is a homogeneous function with degree zero (that is, $\alpha = 0$).

Homogeneous equation (1) is solved by replacing the unknown function $x(t)$ by the function $v(t)$ using the formula

$$x(t) = t \cdot v(t). \tag{2}$$

Since $f(t, x)$ is a homogeneous function with degree zero, we have

$$f(t, x) = f(t, tv) = t^0 \cdot f(1, v) = f(1, v). \tag{3}$$

Differentiating equality (2), we get:

$$x' = v + tv'. \tag{4}$$

Substituting relations (3) and (4) into the original equation (1), we find that the new unknown function $v = v(t)$ satisfies the differential equation

$$tv' = f(1, v) - v.$$

This is an equation with variables separable, and we know how to solve it.

Example 2. Find the general solution of the differential equation

$$x' = \frac{t^2+x^2}{tx}. \tag{5}$$

△ It was shown in Example 1 that the function $f(t, x) = \frac{t^2+x^2}{tx}$ is a homogeneous function with degree zero. Therefore we seek the solution of equation (5) in the form $x = tv$. Then

$$v + tv' = \frac{t^2+t^2v^2}{t^2v} \quad \text{or} \quad tv' = \frac{1+v^2}{v} - v = \frac{1}{v}.$$

Separating the variables in the last equation and integrating, we find

$$\frac{1}{2}v^2 = \ln|t| + C.$$

Hence $v = \pm\sqrt{2(C+\ln|t|)}$, and, finally, $x(t) = tv(t) = \pm t\sqrt{2(C+\ln|t|)}$. ▲

Example 3. Solve the equation

$$y' = \frac{xy}{x^2-y^2}.$$

△ Since

$$f(\mu x, \mu y) = \frac{(\mu x)\cdot(\mu y)}{(\mu x)^2-(\mu y)^2} = \frac{xy}{x^2-y^2} = f(x, y),$$

the differential equation is homogeneous. We seek its solution in the form $y(x) = x\cdot v(x)$. Then

$$v + xv' = \frac{v}{1-v^2} \quad \text{or} \quad x\cdot\frac{dv}{dx} = \frac{v^3}{1-v^2}.$$

We separate the variables:

$$\left(\frac{1}{v^3} - \frac{1}{v}\right)dv = \frac{dx}{x}.$$

Integrating, we find:

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln|C|,$$

where the arbitrary constant taking on all real values is written for conveniency in the form $\ln|C|$. Substituting $v = \frac{y}{x}$, we get the general solution of the original equation in implicit form:

$$-\frac{x^2}{2y^2} = \ln|Cy|.$$

In this case, it is impossible to get y as an explicit function of x written with the help of elementary functions. ▲

3. First-order Linear Equations. The differential equation

$$x' + a(t)\,x = f(t) \tag{1}$$

is called a first-order linear differential equation.

In the preceding subsection, by replacing the unknown function $x(t) = tv(t)$, we reduced the original homogeneous equation to an equation with variables separable with respect to the new unknown function $v(t)$. We also seek the solution of linear equation (1) in the form of a product of two factors:

$$x = uv, \tag{2}$$

where u is some nonzero partial solution of the equation with variables separable

$$u' + a(t)\,u = 0, \tag{3}$$

and v is a new unknown function. Substituting expression (2) into differential equation (1), we get:

$$[u' + a(t)\,u]\,v + uv' = f(t).$$

Then, by virtue of equality (3), we find that the unknown function $v(t)$ will satisfy the equation

$$uv' = f(t). \tag{4}$$

Thus, we have obtained the desired result, since equations (3) and (4) to be solved for finding the functions $u(t)$ and $v(t)$ (and hence, by virtue of formula (2), the function $x(t)$) are differential equations with variables separable. From equations (3) and (4) we find, in succession, the functions u and v, choosing for u a concrete solution other than zero.

Example 1. Find the general solution of the linear differential equation

$$x' + tx = 4t.$$

△ We seek the solution of the given differential equation in the form $x(t) = u(t)\,v(t)$, where $u(t)$ is some partial solution of the equation with variables separable

$$u' + tu = 0. \tag{5}$$

We substitute the expression $x = uv$ into the original equation:

$$v(u' + tu) + uv' = 4t.$$

Hence, bearing in mind equality (5), we find that

$$uv' = 4t. \tag{6}$$

We separate the variables in equation (5) and integrate:

$$u = Ce^{-\frac{t^2}{2}}.$$

For simplicity, let us choose $C = 1$. Then equation (6) takes the form of the simplest differential equation

$$v' = 4te^{\frac{t^2}{2}}.$$

Integrating, we find:

$$v = 4e^{\frac{t^2}{2}} + C.$$

Finally,

$$x = uv = e^{-\frac{t^2}{2}}(4e^{\frac{t^2}{2}} + C) = Ce^{-\frac{t^2}{2}} + 4. \blacktriangle$$

Example 2. Solve the equation

$$x' + tx = \sin t.$$

$\triangle$ The given differential equation is a linear one. We seek its solution in the form $x(t) = u(t)\,v(t)$. We solve, in turn, equations with variables separable (3) and (4):

$$u' + tu = 0 \quad \text{or} \quad \frac{du}{dt} = -tu,$$

whence $u(t) = e^{-\frac{t^2}{2}}$ (see Example 1). Further,

$$v' = \frac{f(t)}{u(t)} = e^{\frac{t^2}{2}} \cdot \sin t,$$

whence

$$v(t) = C + \int_{t_0}^{t} e^{\frac{z^2}{2}} \cdot \sin z\, dz,$$

where C is an arbitrary constant, and t_0 is some fixed number. The obtained integral cannot be expressed in terms of elementary functions. Finally, we get

$$x = uv = Ce^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \int_{t_0}^{t} e^{\frac{z^2}{2}} \sin z\, dz. \blacktriangle \qquad (7)$$

Exercises

1. Solve the following equations with variables separable:

(a) $y' = x + \sin x$; (b) $y' = e^{-y} - 1$; (c) $y' = \frac{y+1}{x+1}$;

(d) $\sqrt{1-t^2}\,x' + tx = 0$; (e) $(\sin x)\,y' = y \ln y$; (f) $y' = e^{x+y}$;

(g) $y \sin x + y' \cos x = 0$; (h) $(1+x)\,y' = 2y$; (i) $(1-y^2) = xy'$.

2. Solve the given homogeneous equations:

(a) $xx' = 2x - t$; (b) $x' = \frac{x}{t} - \frac{t}{x}$; (c) $x^2y' = y^2 + xy$.

3. Find the general solution to each of the given equations:

(a) $y' = \sqrt[3]{y^2}$; (b) $y' = \sqrt{1-y^2}$; (c) $y' = 4x\sqrt{y-1}$;

(d) $xy' + y = y^2$; (e) $\sqrt{y}\sin^2 x + y' = 0$.

4. Find the solution to each of the given Cauchy's problems:

(a) $tx' = 2x$, $x(2) = 3$; (b) $(1-t)x' - x = 0$, $x(0) = 1$;

(c) $x' = \frac{t(1-x^2)}{x(1+t^2)}$, $x(\sqrt{3}) = -\frac{3}{2}$; (d) $x' = \frac{x^2}{t^2} - \frac{x}{t}$, $x(-1) = 1$.

5. Solve the following equations:

(a) $x' = -\frac{2x}{t} + t^2$; (b) $x' = 2t - 2tx$; (c) $x' + x\tan t = \frac{1}{\cos t}$;

(d) $x' = 2tx + (t - t^3)e^{t^2}$;

(e) $y' - \frac{3y}{x} = x$; (f) $y' + \frac{2y}{x} = \frac{e^{-x^2}}{x}$; (g) $xy' + y = \ln x + 1$.

6. Find the solution to each of the given Cauchy's problems:

(a) $(9-t^2)x' + tx = 9$, $x(3) = 3$; (b) $x' - x\tan t = \frac{1}{\cos t}$, $x(0) = 0$;

(c) $x' = x\sin t + 2\sin 2t$, $x(0) = 1$;

(d) $(t+1)x' = 2x + (t+1)^{\frac{1}{2}}$, $x(0) = 2$.

7. An electric circuit consists of a series-connected d.c. source supplying a voltage U, resistor R, self-induction L, and a switch which occupies the ON position when $t = 0$. Set up a differential equation and, solving it, find the dependence of current on time.

8. An electric circuit consists of a series-connected current source whose voltage changes according to the law $U(t) = U_0 \sin \omega t$, resistor R, self-induction L, and a switch which occupies the ON position when $t = 0$. Find the strength of current $I(t)$ in the circuit under steady-state conditions.

11.3. The Methods of Solving Second-order Linear Differential Equations

1. Cauchy's Problem for a Second-order Linear Differential Equation. In the general case, a second-order linear differential equation has the form

$$x'' + a(t)x' + b(t)x = f(t). \tag{1}$$

However, a second-order linear equation cannot be solved in the general form as it was done for a first-order linear equation. Therefore we shall confine ourselves to an important (for applications) particular case of a linear equation with constant coefficients:

$$x'' + ax' + bx = f(t), \tag{2}$$

where a and b are constants. If $f(t) \not\equiv 0$, then equation (2) is called a second-order linear nonhomogeneous equation. And if $f(t) = 0$ for all t under consideration, then equation (2) has the form:

$$x'' + ax' + bx = 0$$

and is called a second-order linear homogeneous equation.

Let us consider a few examples.

Example 1. Write and solve the differential equation of the motion of a material point of mass m under the action of a constant force F.

△ Let the material point move along the x-axis. The coordinate x of the material point is a function of time: $x = x(t)$. The equation of motion is represented by Newton's differential equation:

$$mx''(t) = F,$$

or

$$x''(t) = \frac{F}{m}.$$

Integrating both sides of this equation with respect to t, we get:

$$\int x''(t)\,dt = \int \frac{dx'(t)}{dt}\,dt = \int dx'(t) = x'(t) + C_3,$$

$$\int \frac{F}{m}\,dt = \frac{Ft}{m} + C_4.$$

Thus, we have arrived at the first-order differential equation

$$x'(t) = \frac{Ft}{m} + C_1,$$

where $C_1 = (C_4 - C_3)$ is an arbitrary constant. Integrating both sides of the obtained equation with respect to t, we find the solution of the original second-order equation:

$$x(t) = \frac{Ft^2}{2m} + C_1 t + C_2,$$

where C_1 and C_2 are arbitrary constants. Note that the part of the solution depending on two arbitrary constants C_1 and C_2, that is, the function $x_0 = C_1 t + C_2$, is the solution of the homogeneous equation $x''(t) = 0$ which corresponds to the original nonhomogeneous

equation. Thus,

$$x = x_0 + \frac{Ft^2}{2m}. \blacktriangle$$

Example 2. Write and solve the differential equation of the fall of a material point of mass m in the gravitational field of the Earth.

△ The differential equation describing the change in the velocity of a body freely falling in the gravitational field of the Earth was obtained in 11.1, subsection 1:

$$\frac{dv}{dt} = g - \frac{\rho v}{m},$$

where $v = v(t)$ is the free-fall velocity depending on time, g free-fall acceleration, and ρ the coefficient of air resistance. Let the material point move along the y-axis. The coordinate y of the material point is a function of time: $y = y(t)$. Since $v = y'$, the equation of free fall takes the form:

$$y'' = g - \frac{\rho}{m} y'. \tag{3}$$

Integrating this equation with respect to t, we get a first-order linear differential equation

$$y' = gt - \frac{\rho y}{m} + C,$$

where C is an arbitrary constant. Solving this equation (see subsection 3 of the preceding section), we find:

$$y = C_1 e^{-\frac{\rho t}{m}} + C_2 + \frac{mgt}{\rho}.$$

The part of the solution dependent on two arbitrary constants C_1 and C_2, that is, the function

$$y_0 = C_1 e^{-\frac{\rho t}{m}} + C_2,$$

satisfies the homogeneous equation

$$y'' + \frac{\rho}{m} y' = 0,$$

corresponding to nonhomogeneous equation (3). Thus,

$$y = y_0 + \frac{mgt}{\rho}. \blacktriangle$$

Example 3. Write and solve the differential equation of motion of a material point of mass m under the action of a periodic force $F(t) = F_0 \cdot \sin \omega t$.

△ Let the material point move along the x-axis. The Newton's equation takes the form:

$$mx'' = F_0 \sin \omega t \quad \text{or} \quad x'' = \frac{F_0}{m} \sin \omega t.$$

Integrating this equation with respect to t, we get:

$$x' = -\frac{F_0}{m\omega} \cos \omega t + C_1.$$

Integrating once again with respect to t, we finally find:

$$x = C_1 t + C_2 - \frac{F_0}{m\omega^2} \sin \omega t.$$

Thus,

$$x = x_0 - \frac{F_0}{m\omega^2} \sin \omega t,$$

where the function $x_0 = C_1 t + C_2$, dependent on two arbitrary constants C_1 and C_2, is the solution of the homogeneous equation $x'' = 0$. ▲

Example 4. Solve the differential equation

$$x'' + \frac{k}{m} x = 0,$$

describing the oscillations of a load of mass m on a spring with rigidity k (see 11.1, subsection 1).

△ We denote $\frac{k}{m} = \omega^2$. Then the equation takes the form:

$$x'' + \omega^2 x = 0.$$

Note that the functions $\sin \omega t$ and $\cos \omega t$ are solutions of this equation.

The solution of the equation dependent on two arbirtrary constants will be represented by the function

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t.$$

Indeed,

$$\begin{aligned} x''(t) + \omega^2 x(t) &= (C_1 \sin \omega t + C_2 \cos \omega t)'' + \omega^2 (C_1 \sin \omega t + C_2 \cos \omega t) \\ &= C_1 [(\sin \omega t)'' + \omega^2 \sin \omega t] + C_2 [(\cos \omega t)'' + \omega^2 \cos \omega t] \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \quad ▲ \end{aligned}$$

Let us draw a number of conclusions from these examples. Firstly, we have considered only partial cases of general equation (2) and therefore succeeded in constructing a solution in the first three examples and guessed the solution of the fourth example. Secondly, the solutions of all the examples depend on two arbitrary constants.

Such solutions are called the general solutions of second-order differential equations. Thirdly, the solutions of nonhomogeneous differential equations are the sum of two principally different parts: the first part depends on the arbitrary constants C_1 and C_2 and is the general solution of the corresponding homogeneous equation, while the second part contains no arbitrary constants and is some partial solution of the nonhomogeneous equation.

It turns out that these conclusions hold true for the general case of equation (2) as well. Prior to formulating the statement of the corresponding theorem, let us give the definition of linearly dependent and linearly independent functions.

Definition. Two functions $\varphi_1(t)$ and $\varphi_2(t)$ are said to be linearly dependent if there are numbers λ_1 and λ_2, not equal simultaneously to zero, such that the following identity is valid:

$$\lambda_1\varphi_1(t) + \lambda_2\varphi_2(t) = 0. \tag{4}$$

Otherwise, if we fail to choose such numbers, the functions are called linearly independent. In other words, if the functions φ_1 and φ_2 are linearly independent, and identity (4) occurs, then $\lambda_1 = \lambda_2 = 0$.

It is obvious that the functions φ_1 and φ_2 are linearly dependent if and only if they are proportional to each other.

Example 5. Show that the functions $\varphi_1(t) = e^{k_1 t}$ and $\varphi_2(t) = e^{k_2 t}$ are linearly independent for $k_1 \neq k_2$.

△ Suppose that there is the identity $\lambda_1 \cdot e^{k_1 t} + \lambda_2 \cdot e^{k_2 t} = 0$, where at least one of the coefficients, say λ_2, is nonzero. Then we get the identity

$$e^{(k_2 - k_1)t} = -\frac{\lambda_1}{\lambda_2},$$

which is impossible, since the left-hand side of this equality varies with a change in t, whereas the right-hand side is constant. ▲

Theorem 11.3.1. *Let $\varphi_1(t)$ and $\varphi_2(t)$ be two linearly independent solutions of the homogeneous differential equation corresponding to nonhomogeneous equation* (2). *Then the general solution of equation* (2) *is equal to the sum of the general solution $x_0 = C_1\varphi_1(t) + C_2\varphi_2(t)$ of the corresponding homogeneous equation and partial solution $x_1(t)$ of nonhomogeneous equation* (2), *that is*

$$x(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + x_1(t).$$

Thus, in order to find the general solution of equation (2), we first have to find the general solution $x_0 = C_1\varphi_1(t) + C_2\varphi_2(t)$ of the corresponding homogeneous equation and then to find some partial solution of the nonhomogeneous equation. Hence, all in all we have to find three partial solutions: two different (linearly independent)

partial solutions $\varphi_1(t)$ and $\varphi_2(t)$ of the homogeneous and one partial solution $x_1(t)$ of the nonhomogeneous equation.

How to find the general solution of a homogeneous equation will be explained in the next subsection, and the partial solution of a nonhomogeneous equation will be considered in subsection 3. Now we should like to note that to single out a unique solution of a second-order differential equation, we use the additional initial conditions:

$$x(t_0) = x_0,\ x'(t_0) = v_0. \tag{5}$$

In the case when a second-order differential equation describes mechanical motion, the initial values x_0 and v_0 correspond to the initial coordinate and velocity of motion, respectively.

The problem of finding the solution of equation (2) satisfying initial conditions (5) is called Cauchy's problem for a second-order differential equation. For a wide class of equations (2) Cauchy's problem has a unique solution.

2. The General Solution of a Linear Homogeneous Equation. The second-order homogeneous equation with respect to the unknown function $x = x(t)$

$$x'' + ax' + bx = 0 \tag{1}$$

is equivalent to the system of two first-order linear differential equations with respect to the unknown functions x and v:

$$\begin{cases} x' = v, \\ v' = -av - bx. \end{cases} \tag{2}$$

Instead of the unknown functions x and v, we introduce new unknown functions u_1 and u_2 by the formulas

$$x = u_1 + u_2,\ v = \lambda_1 u_1 + \lambda_2 u_2, \tag{3}$$

where λ_1 and λ_2 are roots of the quadratic equation

$$\lambda^2 + a\lambda + b = 0. \tag{4}$$

From formula (3) we find:

$$(\lambda_2 - \lambda_1)\, u_1 = \lambda_2 x - v,\ (\lambda_2 - \lambda_1)\, u_2 = -\lambda_1 x + v.$$

Differentiating these equalities and taking into account formulas (2)-(4), we get

$$u_1' = \lambda_1 u_1,\quad u_2' = \lambda_2 u_2, \tag{5}$$

where we have also used the equalities $\lambda_1\lambda_2 = b$ and $\lambda_1 + \lambda_2 = -a$. Integrating equations with variables separable (5), we find

$$u_1(t) = C_1 e^{\lambda_1 t},\quad u_2(t) = C_2 e^{\lambda_2 t}.$$

Then, according to formula (3), in the case $\lambda_1 \neq \lambda_2$ the general solution of the second-order homogeneous equation (1) will be represented by the function

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \quad (6)$$

Thus, to find the general solution of the homogeneous equation (1), we have:

(a) to set up the characteristic quadratic equation (4) corresponding to the differential equation (1);

(b) to find the roots λ_1 and λ_2 of this equation;

(c) in the case $\lambda_2 \neq \lambda_1$, to write the general solution of differential equation (1) in the form

$$x(t) = C_1 \cdot e^{\lambda_1 t} + C_2 \cdot e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants. The case $\lambda_2 = \lambda_1$ will be considered later.

Example. Find the general solution of the homogeneous equation

$$x'' + x' - 2x = 0,$$

and also the partial solution satisfying the initial conditions

$$x(0) = 1, \quad x'(0) = 2.$$

△ We set up the characteristic equation:

$$\lambda^2 + \lambda - 2 = 0$$

and find its roots:

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+4 \cdot 2}}{2} = \frac{-1 \pm 3}{2}, \quad \lambda_1 = 1, \ \lambda_2 = -2.$$

The general solution of the original differential equation is

$$x(t) = C_1 e^t + C_2 e^{-2t}.$$

To find the partial solution, we determine C_1 and C_2 from the initial conditions. Since $x'(t) = C_1 e^t - 2C_2 e^{-2t}$, we have

$$\begin{cases} x(0) = C_1 + C_2 = 1, \\ x'(0) = C_1 - 2C_2 = 2, \end{cases}$$

whence $C_1 = \frac{4}{3}$ and $C_2 = -\frac{1}{3}$. Thus, the desired particular solution is represented by $x(t) = \frac{4}{3} e^t - \frac{1}{3} e^{-2t}$. ▲

3. **Real and Complex Roots of the Characteristic Equation.** In the case of a positive discriminant the characteristic equation has two distinct real roots λ_1 and λ_2. This case was already considered in the example of the preceding subsection.

If the discriminant of the characteristic equation is equal to zero, that is, $\sqrt{a^2-4b}=0$, then this equation has only one multiple root $\lambda=-\frac{a}{2}$ (its multiplicity being equal to two), and thus we do not have two linearly independent functions which are necessary to construct the general solution. This case will be considered in the next subsection.

Now let us consider the case when the discriminant is negative, that is, when $(a^2-4b)<0$. If we sought the complex solutions (that is, the functions taking on complex values) of the homogeneous differential equation with real coefficients

$$x''+ax'+bx=0, \tag{1}$$

then no difficulties would appear in this case. Indeed, according to the rule formulated in the preceding subsection, the general complex solution of the real homogeneous differential equation (1) is represented by the function

$$x(t)=C_3e^{(\alpha+i\beta)t}+C_4e^{(\alpha-i\beta)t}, \tag{2}$$

since in this case the roots of the characteristic equation are equal:

$$\lambda_1=\frac{-a+\sqrt{a^2-4b}}{2}=\left(-\frac{a}{2}\right)+i\,\frac{\sqrt{4b-a^2}}{2}=\alpha+i\beta,$$

$$\lambda_2=\frac{-a-\sqrt{a^2-4b}}{2}=\left(-\frac{a}{2}\right)-i\,\frac{\sqrt{4b-a^2}}{2}=\alpha-i\beta,$$

where α and β are real numbers (since $4b-a^2>0$), and $i=\sqrt{-1}$ is the imaginary unit (see 4.4). In formula (2), C_3 and C_4 are arbitrary complex constants.

Since we are interested only in real solutions, we have to single out from the entire set of complex solutions (2) the subset corresponding to the real solutions. For this purpose, we make use of Euler's formula (see 9.2, subsection 5)

$$e^{iy}=\cos y+i\sin y, \tag{3}$$

where y is a real number, and i the imaginary unit. With the help of formula (3), we bring the general complex solution (2) to the form

$$\begin{aligned}x(t)&=e^{\alpha t}\cdot[C_3e^{i\beta t}+C_4e^{-i\beta t}]=e^{\alpha t}\cdot[C_3(\cos\beta t\\&\qquad+i\sin\beta t)+C_4(\cos\beta t-i\sin\beta t)]\\&=e^{\alpha t}\cdot[(C_3+C_4)\cos\beta t+i(C_3-C_4)\sin\beta t]\\&=e^{\alpha t}\cdot[C_1\cos\beta t+C_2\sin\beta t].\end{aligned}$$

It is clear that real solutions correspond to arbitrary real values of the constants C_1 and C_2 (in this case, C_3 and C_4 are complex conjugate). Thus, the general real solution of the homogeneous differential

equation (1) in the case of a negative discriminant of the characteristic equation is represented by the function

$$x(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t, \tag{4}$$

where C_1 and C_2 are arbitrary real constants.

Example 1. Solve the equation

$$x'' + 2x' + 2x = 0.$$

△ We write the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0.$$

It has two complex conjugate solutions: $\lambda_1 = -1 + i$ and $\lambda_2 = \overline{\lambda_1} = -1 - i$.
By formula (4), we find the general solution in which $\alpha = -1$ and $\beta = 1$:

$$x(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t. \ ▲$$

For practical applications an important particular case of solution (4) is the case when the roots of the characteristic equation are imaginary. This happens when in equation (1) the coefficient $a = 0$, and the coefficient $b > 0$. In this case, the characteristic equation takes the form:

$$\lambda^2 + b = 0,$$

whence $\lambda_{1,2} = \pm \sqrt{-b} = \pm i \sqrt{b}$. General solution (4) in this case has the form:

$$x(t) = C_1 \cos (\sqrt{b}t) + C_2 \sin (\sqrt{b}t). \tag{5}$$

Example 2. Find the general solution of the equation

$$x'' + 16x = 0$$

and also the partial solution satisfying the initial conditions: $x(0) = 0$ and $x'(0) = 4$.

△ The characteristic equation $\lambda^2 + 16 = 0$ has two imaginary roots $\lambda_1 = 4i$ and $\lambda_2 = -4i$. The general solution is represented by

$$x(t) = C_1 \cos 4t + C_2 \sin 4t.$$

Let us now find the partial solution for which purpose we first obtain that $x'(t) = -4C_1 \sin 4t + 4C_2 \cos 4t$. The constants C_1 and C_2 are determined from the initial conditions:

$$\begin{cases} x(0) = C_1 = 0, \\ x'(0) = 4C_2 = 4, \end{cases}$$

whence $C_1 = 0$ and $C_2 = 1$. The partial solution is $x(t) = \sin 4t$. ▲

4. The Multiple Root of the Characteristic Equation. In the case when the discriminant $(a^2 - 4b)$ of the characteristic equation is equal to zero, there is only one multiple root λ:

$$\lambda_1 = \lambda_2 = \lambda = -\frac{a}{2}.$$

Formula (6) (see subsection 2) for the general solution of the homogeneous equation

$$x'' + ax' + bx = 0 \tag{1}$$

is incorrect in this case, since there is only one partial solution $\varphi_1(t) = e^{\lambda t}$.

Let us show that in this case the second (missing) partial solution of equation (1) will be expressed by the function $\varphi_2(t) = te^{\lambda t}$ $\left(\text{where } \lambda = -\frac{a}{2}\right)$. Indeed,

$$(te^{\lambda t})' = e^{\lambda t} + \lambda te^{\lambda t}, \quad (te^{\lambda t})'' = (e^{\lambda t} + \lambda te^{\lambda t})' = 2\lambda e^{\lambda t} + \lambda^2 te^{\lambda t}.$$

Then, bearing in mind the equality $\lambda = -\frac{a}{2}$, we find that

$$(te^{\lambda t})'' + a(te^{\lambda t})' + bte^{\lambda t} = (2\lambda + \lambda^2 t + a\lambda t + a + bt)\, e^{\lambda t} = 0.$$

Thus, the general solution of the homogeneous equation (1) in the case of the multiple root λ of the characteristic equation is expressed by the function

$$x(t) = C_1 e^{\lambda t} + C_2 te^{\lambda t}, \text{ where } \lambda = -\frac{a}{2}. \tag{2}$$

Example. Find the general solution of the equation

$$x'' - 6x' + 9x = 0.$$

△ The characteristic equation

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

has one multiple root $\lambda = 3$. According to formula (2), the general solution of the given differential equation will be represented by

$$x(t) = C_1 e^{3t} + C_2 te^{3t}. \ \blacktriangle$$

5. The Partial Solution of a Linear Nonhomogeneous Equation in the Case of Quasi-polynomials. As we already know, to find the general solution of the nonhomogeneous equation

$$x'' + ax' + bx = f(t), \tag{1}$$

we have to find the general solution of the corresponding homogeneous equation

$$x'' + ax' + bx = 0 \tag{2}$$

and one partial solution of nonhomogeneous equation (1).

The first of these problems was solved in the previous subsections. Let us now consider the problem on finding the partial solution of nonhomogeneous equation (1). We shall confine ourselves to the particular case which is of importance for technical applications, namely, to the case when the function $f(t)$ is a quasi-polynomial:

$$f(t) = P_m(t)\, e^{\gamma t}, \tag{3}$$

that is, the product of a given polynomial of degree m by an exponential function with a complex (in the general case) constant γ.

We can show that the partial solution of equation (1) should be sought in one of the following three forms:

(a) if γ coincides with neiher of the two distinct roots λ_1 and λ_2 of the characteristic equation, then we seek the partial solution $x_1(t)$ in the form

$$x_1(t) = Q_m(t)\, e^{\gamma t}, \tag{4}$$

where $Q_m(t)$ is a standard polynomial of degree m with unknown coefficients;

(b) if γ coincides with one of the two distinct roots λ_1 and λ_2 of the characteristic equation, then

$$x_1(t) = tQ_m(t)\, e^{\gamma t}; \tag{5}$$

(c) if γ coincides with the multiple root λ of the characteristic equation, then

$$x_1(t) = t^2 Q_m(t)\, e^{\gamma t}. \tag{6}$$

The following are concrete examples illustrating how to find unknown coefficients in polynomials Q_m.

Example 1. Find the general solution of the equation

$$x'' + 2x' - 3x = 12. \tag{7}$$

△ The characteristic equation $\lambda^2 + 2\lambda - 3 = 0$ has two roots $\lambda_1 = -3$ and $\lambda_2 = 1$. The general solution of the homogeneous equation is $x_0(t) = C_1 e^{-3t} + C_2 e^t$. Since the right-hand side of the given nonhomogeneous differential equation may be regarded as a quasi-polynomial: $P_0(t)e^{0 \cdot t} = 12 \cdot e^{0 \cdot t} = 12$, we seek the partial solution in the form (4), that is, in the form of a polynomial of zero degree:

$$x_1(t) = Q_0(t) = A_0,$$

where A_0 is yet unknown coefficient. Substituting this solution into original equation (7), we get

$$-3A_0 = 12.$$

Hence we find $A_0 = -4$. Thus, the general solution of equation (7) is

$$x(t) = C_1 e^{-3t} + C_2 e^t - 4. \blacktriangle$$

Example 2. Find the general solution of the equation

$$x'' + 2x' - 3x = 6t.$$

$\triangle$ The general solution of the homogeneous equation is $x_0(t) = C_1 e^{-3t} + C_2 e^t$. Since here $\gamma = 0 \neq \lambda_1, \lambda_2$, we seek the partial solution of the nonhomogeneous equation in form (4), that is, in the form of some polynomial of first degree:

$$x_1(t) = Q_1(t) = A_0 t + A_1,$$

where A_0 and A_1 are unknown coefficients. Substituting this function into the original equation, we get

$$-3A_0 t + 2A_0 - 3A_1 = 6t.$$

Since this equality must be satisfied for any t, we have to equate the coefficients of like powers of t:

$$\begin{cases} -3A_0 = 6, \\ 2A_0 - 3A_1 = 0, \end{cases}$$

whence $A_0 = -2$, $A_1 = -\frac{4}{3}$. Finally, the sought-for general solution is

$$x(t) = C_1 e^{-3t} + C_2 e^t - 2t - \frac{4}{3}. \blacktriangle$$

Example 3. Find the general solution of the equation

$$x'' + 9x = (t^2 + 2) e^{3t}.$$

$\triangle$ The characteristic equation $\lambda^2 + 9 = 0$ has two imaginary roots $\lambda_1 = 3i$ and $\lambda_2 = -3i$, that is, the general solution of the homogeneous equation is

$$x_0(t) = C_1 \cos 3t + C_2 \sin 3t.$$

Since the right-hand side of the given equation $(t^2 + 2)e^{3t}$ has the form $P_2(t) e^{3t}$, we seek the partial solution of the nonhomogeneous equation in form (4), that is, in the form of the product of a polynomial of second degree by the exponential function:

$$x_1(t) = Q_2(t) e^{3t} = (A_0 t^2 + A_1 t + A_2) e^{3t},$$

where A_0, A_1, and A_2 are unknown coefficients. Substituting this function into the original differential equation and collecting like

terms, we get:

$$[18A_0t^2 + (18A_1 + 12A_0)\,t + (18A_2 + 6A_1 + 2A_0)]e^{3t} = [1\cdot t^2 + 0\cdot t + 2]e^{3t}.$$

We equate the coefficients of like powers of t:

$$\begin{cases} 18A_0 = 1, \\ 18A_1 + 12A_0 = 0, \\ 18A_2 + 6A_1 + 2A_0 = 2, \end{cases}$$

whence $A_0 = \frac{1}{18}$, $A_1 = -\frac{1}{27}$, and $A_2 = \frac{19}{162}$. Finally, the sought-for general solution is

$$x(t) = C_1 \cos 3t + C_2 \sin 3t + \left(\frac{t^2}{18} - \frac{t}{27} + \frac{19}{162}\right) e^{3t}. \blacktriangle$$

Example 4. Find the general solution of the equation

$$x'' - 4x = 5e^{-2t}.$$

$\triangle$ The characteristic equation $\lambda^2 - 4 = 0$ has the roots $\lambda_1 = 2$ and $\lambda_2 = -2$, that is, we get the general solution of the homogeneous equation:

$$x_0(t) = C_1e^{2t} + C_2e^{-2t}.$$

Since the right-hand side of the given equation $5e^{-2t}$ has the form $P_0(t)\,e^{\gamma t}$, and $\gamma = -2 = \lambda_2$, we seek the partial solution of the nonhomogeneous equation in form (5), that is, in the form of the product of t by a polynomial of degree zero and by the exponential function:

$$x_1(t) = tQ_0(t)\,e^{-2t} = A_0te^{-2t},$$

where A_0 is a coefficient, yet unknown. Substituting this function into the original differential equation, we get:

$$-4A_0e^{-2t} + 4A_0te^{-2t} - 4A_0te^{-2t} = 5e^{-2t},$$

whence $A_0 = -\frac{5}{4}$. Thus, the desired general solution is

$$x(t) = C_1e^{2t} + C_2e^{-2t} - \frac{5}{4}te^{-2t}. \blacktriangle$$

Example 5. Find the general solution of the equation

$$x'' + 2x' + 5x = 3\sin t. \tag{8}$$

$\triangle$ The characteristic equation $\lambda^2 + 2\lambda + 5 = 0$ has the complex roots $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$ so that the general solution of the homogeneous equation is

$$x_0(t) = C_1e^{-t}\cos 2t + C_2e^{-t}\sin 2t.$$

We then find the partial solution $x_1(t) = u(t) + iv(t)$ of the equation

$$x'' + 2x' + 5x = 3e^{it}. \tag{9}$$

since the imaginary part of this solution satisfies the original equation (8). Indeed, if

$$(u + iv)'' + 2(u + iv)' + 5(u + iv) = 3\cos t + 3i \sin t,$$

then

$$v'' + 2v' + 5v = 3 \sin t.$$

We seek the partial solution of equation (9) in the form

$$x_1(t) = A_0 e^{it},$$

since $\gamma = i \neq \lambda_1, \lambda_2$. Substituting this function into equation (9), we get:

$$(4 + 2i) A_0 = 3 \text{ or } A_0 = \frac{3}{4+2i} = \frac{6-3i}{10}.$$

Thus,

$$x_1(t) = \frac{6-3i}{10}(\cos t + i \sin t) = \frac{3}{5}\cos t + \frac{3}{10}\sin t + i\left(\frac{3}{5}\sin t - \frac{3}{10}\cos t\right).$$

The imaginary part of this function is the partial solution of original equation (8). Finally, we get the desired general solution

$$x(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + \frac{3}{5}\sin t - \frac{3}{10}\cos t. \blacktriangle$$

Exercises

1. Solve the following equations:

(a) $x'' - 2x' = 0$; (b) $x'' + 5x' + 6x = 0$;

(c) $3x'' - 2x' - 8x = 0$;

(d) $x'' + 4x' + 13x = 0$; (e) $x'' + x' + x = 0$;

(f) $x'' - 6x' + 9x = 0$;

(g) $x'' - x = 2$; (h) $x'' + 2x' = 2$; (i) $x'' + 9x = 9$;

(j) $x'' - 2x' - 3x = e^{4t}$; (k) $x'' - 3x' + 2x = \sin t$; (l) $x'' + x = 4 \sin t$.

2. Find the solution to each of the given Cauchy's problems:

(a) $x'' - 3x' + 2x = 0$, $x(0) = 2$, $x'(0) = 3$;
(b) $x'' + 2x' + 5x = 0$, $x(0) = x'(0) = 1$;
(c) $x'' + 6x' + 9x = 0$, $x(0) = 2$, $x'(0) = 1$;
(d) $x'' + 2x' + 2x = 0$, $x(0) = A$, $x'(0) = B$;
(e) $x'' - x' - 6x = 2$, $x(0) = 1$, $x'(0) = 0$;
(f) $x'' - 9x' = 2 - t$, $x(0) = 0$, $x'(0) = 1$;
(g) $x'' + 4x = 2\cos 2t$, $x(0) = 0$, $x(0) = 4$.

3. For what values of a and b are all the solutions $x = x(t)$ of the equation

$$x'' + ax' + bx = 0$$

periodic functions?

4. For what values of a does the equation

$$x'' + ax = 0$$

have solutions satisfying the conditions $x(0) = x(\pi) = 0$?

5. Solve the following equations:

(a) $x'' - 2x' + x = e^{2t}$; (b) $x'' - 4x = 8t^3$; (c) $x'' + 3x' = 9t$;
(d) $x'' - 3x' + 2x = e^t$; (e) $x'' - 2x' = t^2 - t$; (f) $x'' + 5x' + 6x = e^{-t} + e^{-2t}$.

11.4. The Differential Equation of Oscillations

1. Problems That Lead to the Differential Equation of Oscillations. In subsection 1 of 11.1 we got the differential equation of oscillations of a load fixed on a spring:

$$x'' + \frac{k}{m} \cdot x = 0, \tag{1}$$

where m is the mass of the load, k the spring force, and $x = x(t)$ the desired function describing the dependence of the coordinate x of the load on time t (see Fig. 135). If the load were also acted upon by the resistance force of the surrounding medium, then (see subsection 1 of 11.1) the equation of oscillations would take the form:

$$x'' + \frac{\rho}{m} \cdot x' + \frac{k}{m} \cdot x = 0, \tag{2}$$

where ρ is the coefficient characterizing the resistance force of the medium.

If, in addition to the already mentioned forces, the load were acted upon by some external swinging force $F(t)$, then the equation of

oscillations of the load would take the most general form:

$$x'' + \frac{\rho}{m} \cdot x' + \frac{k}{m} \cdot x = \frac{F(t)}{m}. \tag{3}$$

It is clear that equations (1) and (2) are particular cases of equation of oscillations (3): equation (1) corresponds to the case of absence of external action and forces of friction (resistance), and equation (2) corresponds to the case of absence of external action only. Both of these equations are called the equations of free oscillations (with or without friction).

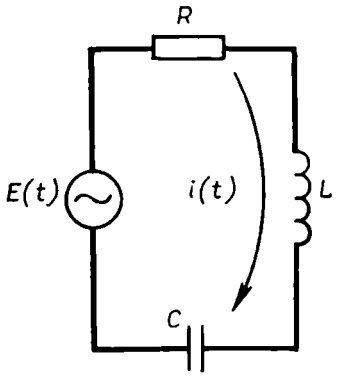

Fig. 136

Let us now consider the problem on alternating current $i(t)$ flowing in a closed electric circuit whose diagram is shown in Fig. 136. The circuit consists of a source (generator) of alternating e.m.f. $E(t)$, a resistor with resistance R, an inductance coil with self-induction L, and a capacitor with capacity C. As is known from electrical engineering, the voltage drop across a resistor is equal to $U_R = Ri(t)$, on the inductance coil it is equal to $U_L = L\frac{di(t)}{dt}$, and on the capacitor to $U_C = \frac{q(t)}{C}$, where q is the capacitor charge. In this case, the quantity of current in the circuit is equal to the rate of change in the capacitor charge, that is, $i(t) = \frac{dq(t)}{dt}$. In conclusion, let us write Ohm's law for a closed circuit

$$Ri + L\frac{di}{dt} + \frac{q}{C} = E(t),$$

that is, the sum of voltage drops along the circuit is equal to the e.m.f. of the source.

Since $i = q'$, this equation takes the form of the second-order differential equation with respect to the unknown function $q = q(t)$:

$$q'' + \frac{R}{L}q' + \frac{q}{LC} = \frac{E(t)}{L}. \tag{4}$$

It is obvious that the equation of electrical oscillations (4) of a charge is completely similar to the equation of mechanical oscillations of a load fixed on spring, and both of these equations can be written in the form

$$x'' + ax' + bx = f(t), \tag{5}$$

that is, in the general form of a second-order linear differential equation with constant coefficients considered in the preceding section. In oscillation theory studied in physics the coefficients a and b

in equation (5) are customarily denoted as follows: $a = \delta$ and $b = \omega_0^2$. Thus, the general equation of oscillations has the form:

$$x'' + \delta x' + \omega_0^2 x = f(t). \tag{6}$$

Here $x(t)$ is the sought-for function describing an oscillating process; its concrete physical meaning may be different (the coordinate of a load fixed on a spring, the capacitor charge in the electric circuit represented in Fig. 136, and a variety of other examples). The coefficient δ is usually called the oscillation damping factor, the coefficient ω_0 the natural frequency, and $f(t)$ constraining force.

2. Solving Cauchy's Problem for the Equation of Oscillations. Cauchy's problem for the generalized equation of oscillations

$$x'' + \delta x' + \omega_0^2 x(t) = f(t) \tag{1}$$

presupposes the specification of the initial conditions:

$$x(0) = x_0, \quad x'(0) = v_0, \tag{2}$$

called, by mechanical analogy, the initial deviation (coordinate, position) and the initial velocity, respectively.

A various set of quasi-polynomials on the right-hand side of equation (1) for which we know how to find the solution of the differential equation in the general form well corresponds to many practical realizations of the constraining force $f(t)$. The most important case is the case of harmonic constraining force

$$f(t) = F_0 \sin \omega t, \tag{3}$$

where F_0 is called the amplitude of the constraining force, and ω its frequency.

Let us find the general solution of oscillation equation (1) in the case of a harmonic constraining force (3). First, we find the general solution of the homogeneous equation for which purpose we write its characteristic equation:

$$\lambda + \delta\lambda + \omega_0^2 = 0.$$

The roots of the characteristic equation are

$$\lambda_1 = \frac{-\delta + \sqrt{\delta^2 - 4\omega_0^2}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\delta - \sqrt{\delta^2 - 4\omega_0^2}}{2}, \tag{4}$$

and the general solution is

$$x_0 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \tag{5}$$

Now we find the partial solution of the homogeneous equation by representing its right-hand side as a sum of two complex quasipolynomials:

$$f(t) = F_0 \sin \omega t = \left(\frac{F_0}{2i}\right) e^{i\omega t} - \left(\frac{F_0}{2i}\right) e^{-i\omega t}.$$

We seek the partial solution of the nonhomogeneous equation $x_1(t)$ in the form

$$x_1(t) = \varphi_1(t) + \varphi_2(t) = A_0 e^{i\omega t} + B_0 e^{-i\omega t}.$$

Substituting this function into the original differential equation, we find:

$$(-\omega^2 + i\delta\omega + \omega_0^2) A_0 e^{i\omega t} + (-\omega^2 - i\delta\omega + \omega_0^2) B_0 e^{-i\omega t} = \left(\frac{F_0}{2i}\right) e^{i\omega t} - \left(\frac{F_0}{2i}\right) e^{-i\omega t}.$$

Equating the coefficients of like functions on the left- and right-hand sides of this equality and performing identity transformations, we get:

$$A_0 = \frac{F_0}{2}\left[\frac{i(\omega^2-\omega_0^2)-\omega\delta}{(\omega^2-\omega_0^2)^2+(\omega\delta)^2}\right], \quad B_0 = \frac{F_0}{2}\left[\frac{-i(\omega^2-\omega_0^2)-\omega\delta}{(\omega^2-\omega_0^2)^2+(\omega\delta)^2}\right].$$

Finally, the general solution of the equation of oscillations (1) is:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \left[\frac{\omega^2-\omega_0^2}{(\omega^2-\omega_0^2)^2+(\omega\delta)^2}\right] F_0 \sin\omega t - \left[\frac{\omega\delta}{(\omega^2-\omega_0^2)^2+(\omega\delta)^2}\right] F_0 \cos\omega t, \qquad (6)$$

where we have made use of the relations $e^{i\omega t} + e^{-i\omega t} = 2\cos\omega t$ and $e^{i\omega t} - e^{-i\omega t} = 2i\sin\omega t$ that follow from Euler s formula (see formula (3) in subsection 5 of 9.2).

Cauchy's problem will be solved if, with the help of formula (6) and initial conditions (2), we find the arbitrary constants C_1 and C_2. However, this will lead to awkward formulas, and therefore we shall not write out the solution of Cauchy's problem in the explicit form.

3. Free Oscillations. In the preceding subsection we obtained an awkward general solution of the equation of oscillations. To study various characteristic features of an oscillating process, let us consider separately the general solution of the homogeneous equation and partial solution of the nonhomogeneous equation.

As we already noted, the homogeneous equation and its solution describe the so-called free oscillations (see formula (5) in the preceding subsection):

$$x_0 = e^{-\frac{\delta}{2}t}\left[C_1 e^{\sqrt{\left(\frac{\delta}{2}\right)^2-\omega_0^2}\,t} + C_2 e^{-\sqrt{\left(\frac{\delta}{2}\right)^2-\omega_0^2}\,t}\right], \qquad (1)$$

where the arbitrary constants C_1 and C_2 must be determined from the initial conditions. These constants, however, do not influence considerably the character of free oscillations.

For $\delta = 0$ free oscillations are harmonic periodic oscillations with frequency ω_0 which is therefore called the frequency of free oscilla-

tions:

$$x_0 = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} = (C_1 + C_2)\cos\omega_0 t + (iC_1 - iC_2)\sin\omega_0 t$$
$$= C_3 \cos\omega_0 t + C_4 \sin\omega_0 t, \qquad (2)$$

where C_3 and C_4 are real arbitrary constants.

For $\delta \neq 0$ free oscillations are damped oscillations since it follows from formula (1) that

$$\lim_{t\to+\infty} x_0 = 0.$$

If the damping factor δ is small so that $\left(\frac{\delta}{2}\right)^2 - \omega_0^2 < 0$, then free oscillations are described by the product of the periodic harmonic function by the exponential function $e^{-\frac{\delta t}{2}}$:

$$x_0 = (C_3 \cos\omega_1 t + C_4 \sin\omega_1 t)\, e^{-\frac{\delta t}{2}}, \qquad (3)$$

where $\omega_1 = \sqrt{\omega_0^2 - \left(\frac{\delta}{2}\right)^2}$ is the frequency depending on the damping factor δ.

For $\left(\frac{\delta}{2}\right)^2 - \omega_0^2 > 0$, that is, for sufficiently large values of the damping factor δ solution (1) corresponds to aperiodic processes since in this case the exponential functions

$$e^{\pm\sqrt{\left(\frac{\delta}{2}\right)^2 - \omega_0^2}\, t}$$

are monotonic functions.

Figure 137 represents the graphs of the function $x = x(t)$ corresponding to the three cases described above.

4. Forced Oscillations. The Amplitude-frequency Characteristic and Resonance. If $\delta = 0$, then free oscillations damp and in a sufficiently long time elapsed from the beginning of oscillations become negligibly small as compared with harmonic undamped forced oscillations corresponding to the partial solution of the nonhomogeneous equation:

$$x_1(t) = \left[\frac{\omega_0^2 - \omega^2}{(\omega^2 - \omega_0^2)^2 + (\omega\delta)^2}\right] F_0 \sin\omega t - \left[\frac{\omega\delta}{(\omega^2 - \omega_0^2)^2 + (\omega\delta)^2}\right] F_0 \cos\omega t, \qquad (1)$$

where F_0 is the amplitude of the constraining force, ω its frequency, ω_0 the frequency of free oscillations. We write equality (1) in the form

$$x_1(t) = B\sin(\omega t - \psi), \qquad (2)$$

where

$$B = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\omega\delta)^2}}, \quad \psi = \arctan\left(\frac{\omega\delta}{\omega^2 - \omega_0^2}\right). \qquad (3)$$

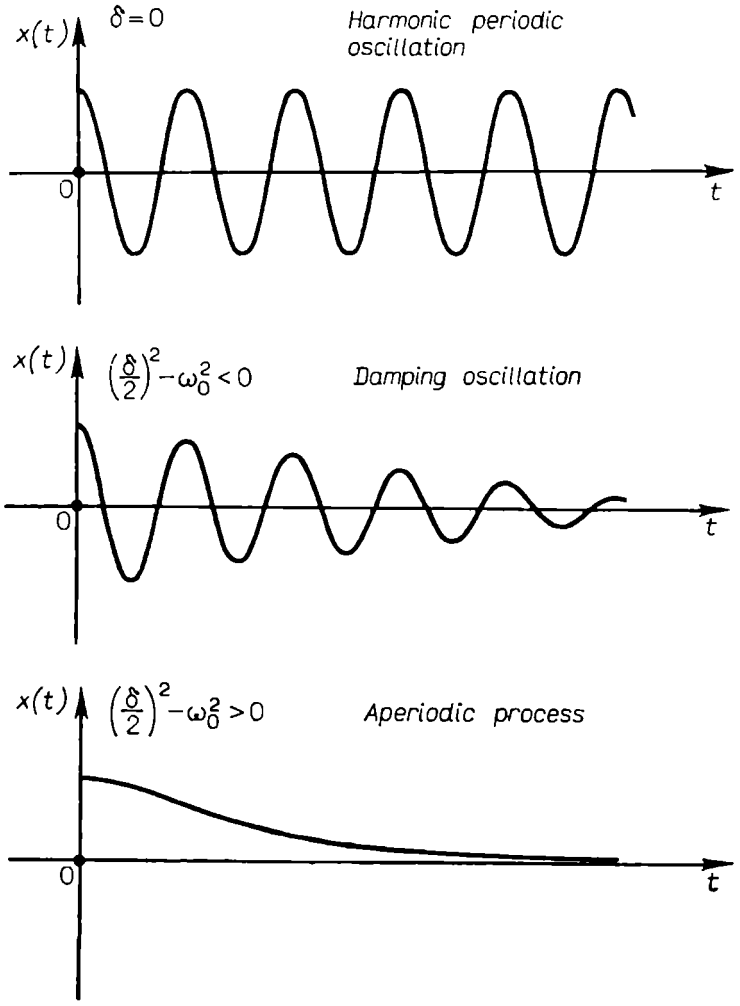

Fig. 137

The forced oscillation amplitude B depends on the frequency of the constraining force ω. This dependence is called the amplitude-frequency characteristic of an oscillation process. As is seen from formula (3), for small δ the amplitude of forced oscillations increases sharply as $\omega \to \omega_0$. This phenomenon is called resonance. It is widely used in practice: if you want to strongly build up an oscillating system by a small force, then the build-up must be realized with a frequency close to that of natural oscillations of the system. This phenomenon should also be taken into consideration when such a build-up is undesirable; to this end, one has to change the characteristics of the construction so that its natural frequency is kept away as far as possible from possible frequencies of external actions on the construction under consideration.

Exercises

1. Find the equation of harmonic oscillations which is satisfied by the function $x = \sin t + \cos t$.

2. Which of the given functions describes a harmonic oscillatory process:

(a) $x = \sin t + \cos t$; (b) $x = t \sin t$; (c) $x = \sin t + \cos 2t$;

(d) $x = \cos t + 1$; (e) $x = e^{-2t} \cdot \sin 3t$; (f) $x = 4e^{-3t} + 2e^{-t}$?

3. Two equal loads are suspended from the end of a spring. Find the motion of one of the loads if the other falls down. The elongation of the spring under the action of one load at the state of rest is a, and free fall acceleration is g. Assume that the load fell down at the time instant $t = 0$.

4. An oscillatory circuit consists of a capacitor whose capacitance $C = 405$ nF, a coil whose inductance $L = 10$ mH, and a resistor whose resistance $R = 2$ Ohms. How many times will the potential difference across the capacitor plates decrease during one period of oscillations?

Chapter 12

ELEMENTS OF INTEGRAL CALCULUS: FUNCTIONS OF SEVERAL VARIABLES

12.1. Double Integrals

1. The definition of a Double Integral (the Case of a Rectangle). Let a function $f(x, y)$ be defined on the rectangle D: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ (Fig. 138).

We divide the interval $[a, b]$ of the x-axis by the points

$$x_i = a + \frac{b-a}{n} i, \quad i = 0, 1, \ldots, n,$$

into n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, of length $\Delta x_i = \frac{b-a}{n}$, and the interval $[c, d]$ of the y-axis by the points

$$y_j = c + \frac{d-c}{n} j, \quad j = 0, 1, \ldots, n,$$

into n subintervals $[y_{j-1}, y_j]$, $j = 1, 2, \ldots, n$, of length $\Delta y_j = \frac{d-c}{n}$.

Then the rectangle D will be partitioned into n^2 rectangles D_{ij} $(i, j = 1, 2, \ldots, n)$, the area of each of which being equal to $\Delta x_i \Delta y_j$.

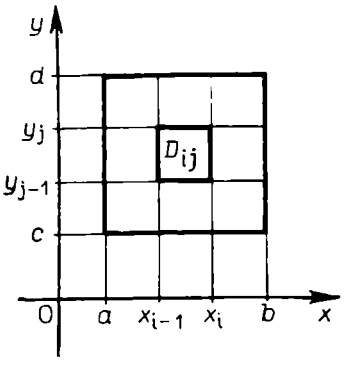

Fig. 138

As in the case of a function of one variable, we form an integral sum for the function of two variables $f(x, y)$:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(\tau_i, t_j) \Delta x_i \Delta y_j, \qquad (1)$$

where (τ_i, t_j) is some point of the rectangle D_{ij}. Sum (1) is called the integral sum of the function $f(x, y)$ corresponding to a given partition of the rectangle D.

If the limit of integral sum (1) as $n \to \infty$ exists and is independent of the choice of points (τ_i, t_j), then it is called the double integral of the function $f(x, y)$ over the rectangle D and is denoted by

$$\iint_D f(x, y)\, dx\, dy.$$

Thus, by definition,

$$\iint\limits_D f(x,\ y)\,dx\,dy=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\tau_i,\ t_j)\,\Delta x_i\,\Delta y_j.$$

It is obvious that the double integral of the function $f(x, y) = 1$ over the rectangle D is equal to the area of the rectangle D, that is

$$\iint\limits_D dx\,dy=(b-a)(d-c).$$

Theorem 12.1.1. *If the double integrals of the functions $f(x, y)$ and $g(x, y)$ over the rectangle D exist, then for any numbers A and B the double integral over the rectangle D of the function $Af(x, y) + Bg(x, y)$ exists, and the following formula is valid:*

$$\iint\limits_D (Af(x,\ y)+Bg(x,\ y))\,dx\,dy=A\iint\limits_D f(x,\ y)\,dx\,dy$$
$$+B\iint\limits_D g(x,\ y)\,dx\,dy.$$

□ According to the definition of the double integral and the properties of limits of sequences, we have:

$$\iint\limits_D (Af(x,\ y)+Bg(x,\ y))\,dx\,dy$$
$$=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}(Af(\tau_i,\ t_j)+Bg(\tau_i,\ t_j))\,\Delta x_i\,\Delta y_j$$
$$=A\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\tau_i,\ t_j)\,\Delta x_i\,\Delta y_j+B\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} g(\tau_i,\ t_j)\,\Delta x_i\,\Delta y_j$$
$$=A\iint\limits_D f(x,\ y)\,dx\,dy+B\iint\limits_D g(x,\ y)\,dx\,dy.\ \blacksquare$$

From the above theorem it follows that a constant factor can be taken outside the sign of a double integral, and the double integral of the sum (difference) of two functions is equal to the sum (difference) of the double integrals of these functions.

Theorem 12.1.2. *If a function $f(x, y)$ $(x, y) \in D$: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ is continuous, then it is integrable on D, and*

$$\iint\limits_D f(x,\ y)\,dx\,dy=\int_a^b\left(\int_c^d f(x,\ y)\,dy\right)dx. \qquad (2)$$

□ Let us prove the validity of formula (2). We divide the closed interval $[a, b]$ on the x-axis by points x_i, $i = 0, 1, \ldots, n$, into n equal subintervals $[x_{i-1}, x_i]$. Then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(\int_c^d f(x, y)\, dy \right) dx.$$

By the mean-value theorem for integrals, there is $\tau_i \in [x_{i-1}; x_i]$, such that

$$\int_{x_{i-1}}^{x_i} \left(\int_c^d f(x, y)\, dy \right) dx = \left(\int_c^d f(\tau_i, y)\, dy \right) \Delta x_i.$$

Then we divide the segment $[c, d]$ of the y-axis by points y_j, $j = 0, 1, \ldots, n$, into n equal subintervals $[y_{j-1}, y_j]$. Then

$$\int_c^d f(\tau_i, y)\, dy = \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} f(\tau_i, y)\, dy.$$

By the mean-value theorem for integrals, there is $t_j \in [y_{j-1}, y_j]$ such that

$$\int_{y_{j-1}}^{y_j} f(\tau_i, y)\, dy = f(\tau_i, t_j)\, \Delta y_j.$$

As a result, we get the equality

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \sum_{i=1}^{n} \sum_{j=1}^{n} f(\tau_i, t_j)\, \Delta x_i\, \Delta y_j.$$

Passing to the limit in this equality as $n \to \infty$, we get formula (2). ■

Formula (2) is known as the formula for reducing a double integral to an iterated one.

Usually, the iterated integral on the right-hand side of formula (2) is written in a different way:

$$\iint_D f(x, y)\, dx\, dy = \int_a^b dx \int_c^d f(x, y)\, dy. \tag{3}$$

Similarly to this formula, we prove the validity of the other formula for reducing a double integral to an iterated one:

$$\iint_D f(x, y)\, dx\, dy = \int_c^d dy \int_a^b f(x, y)\, dx. \tag{4}$$

Theorem 12.1.2 points out a sufficiently wide class of integrable functions and provides the method for computing double integrals.

Example 1. Evaluate the double integral of the function $f(x, y) = 3 - x - xy^2$ over the rectangle D: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 3$.

△ For evaluating the double integral, we make use of the formula for reducing a double integral to an interated one. We then have:

$$\iint_D (3-x-xy)^2\,dx\,dy = \int_0^1 dx \int_0^3 (3-x-xy^2)\,dy.$$

First, assuming x to be constant, we compute the integral with respect to y:

$$\int_0^3 (3-x-xy^2)\,dy = \left(3y - xy - \frac{xy^3}{3}\right)\Big|_0^3 = 9-12x.$$

Then we compute the integral of the obtained function with respect to x.

Then

$$\iint_D (3-x-xy^2)\,dx\,dy = \int_0^1 (9-12x)\,dx = (9x-6x^2)\Big|_0^1 = 3. \ \blacktriangle$$

Example 2. Evaluate the double integral of the function $f(x, y) = e^y \cos x + ye^x$ over the rectangle D: $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant 2$.

△ To evaluate the double integral, we make use of the formula for reducing a double integral to an iterated one. Then, we have:

$$\begin{aligned}\iint_D (e^y \cos x + ye^x)\,dx\,dy &= \int_0^2 dy \int_0^\pi (e^y \cos x + ye^x)\,dx \\ &= \int_0^2 (e^y \sin x + ye^x)\Big|_0^\pi dy \\ &= \int_0^2 y(e^\pi - 1)\,dy = (e^\pi - 1)\frac{y^2}{2}\Big|_0^2 \\ &= 2(e^\pi - 1). \ \blacktriangle\end{aligned}$$

2. Double Integral (the case of an arbitrary domain). Suppose we are given a function $f(x, y)$ defined in some bounded domain G on the coordinate plane. Let D denote the least rectangle with sides parallel to the coordinate axes (Fig. 139) containing the domain G.

Consider the function $f_0(x, y)$ which coincides with $f(x, y)$ at points $(x, y) \in G$ and is equal to zero outside G.

Further, by definition, we shall assume that the following equality is valid

$$\iint\limits_G f(x,\ y)\,dx\,dy = \iint\limits_D f_0(x,\ y)\,dx\,dy. \tag{1}$$

Let us now consider the case when G is the domain bounded from the left and from the right by segments of the straight lines $x = a$ and $x = b$, and from below and above by the graphs of continuous

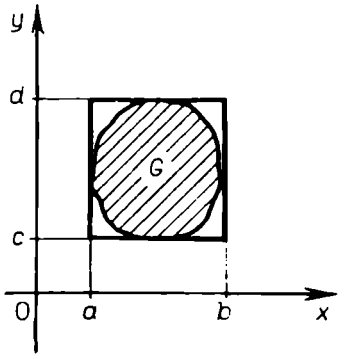

Fig. 139

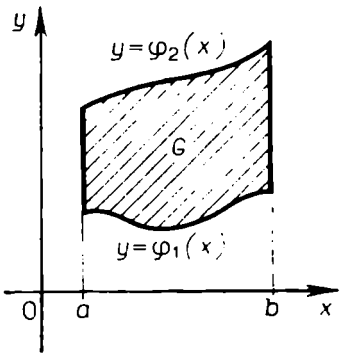

Fig. 140

functions $y = \varphi_1(x)$ and $y = \varphi_2(x)$ (see Fig. 140), that is, G: $a \leqslant x \leqslant b$, $\varphi_1(x) \leqslant y \leqslant \varphi_2(x)$. For such a domain formula (1) takes the form:

$$\iint\limits_G f(x,\ y)\,dx\,dy = \int\limits_a^b dx \int\limits_c^d f_0(x,\ y)\,dy. \tag{2}$$

Since, by definition, with x fixed the function $f_0(x, y)$ coincides with $f(x, y)$ on the closed interval $[\varphi_1(x), \varphi_2(x)]$ and is equal to zero outside this interval, formula (2) can be written as follows:

$$\iint\limits_G f(x,\ y)\,dx\,dy = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy. \tag{3}$$

Similarly, for the case of the domain G:

$$\varphi_1(y) \leqslant x \leqslant \varphi_2(y), \quad c \leqslant y \leqslant d$$

(that is, the domain bounded from below and above by segments of the straight lines $y = c$ and $y = d$, and from left and right by the graphs of continuous functions $x = \varphi_1(y)$ and $x = \varphi_2(y)$; see Fig.

141) we have the following formula for computing the double integral of $f(x, y)$ over G:

$$\iint_G f(x, y)\,dx\,dy = \int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x, y)\,dx. \tag{4}$$

Formulas (3) and (4), as formulas (3) and (4) in the preceding subsection, are referred to as formulas for reducing a double integral to an iterated one.

Example. Evaluate the double integral

$$\iint_G (3xy^2 + x^2)\,dx\,dy,$$

where G is the domain bounded by the lines $y = x^2$, $y = 0$, and $x = 1$ (Fig. 142).

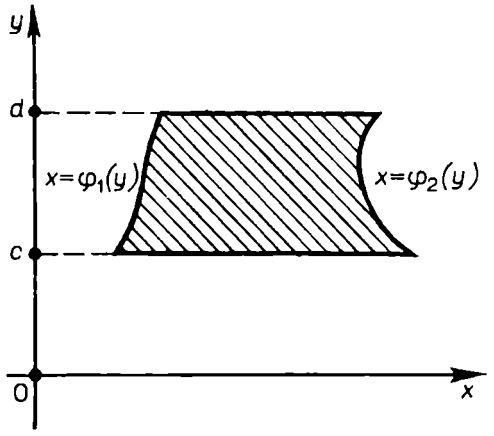

Fig. 141

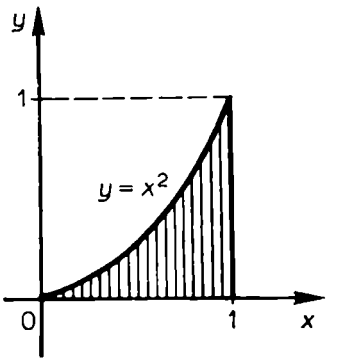

Fig. 142

△ By hypothesis, the domain of integration is the domain G: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant x^2$. To evaluate the double integral over the domain G, we make use of formula (3):

$$\iint_G (3xy^2 + x^2)\,dx\,dy = \int_0^1 dx \int_0^{x^2} (3xy^2 + x^2)\,dy = \int_0^1 (xy^3 + x^2y)\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 (x^7 + x^4)\,dx = \left(\frac{x^8}{8} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{13}{40}.$$

Note. For the same purpose, we can use formula (4) as well:

$$\iint\limits_G (3xy^2 + x^2)\,dx\,dy = \int\limits_0^1 dy \int\limits_{\sqrt{y}}^1 (3xy^2 + x^2)\,dx$$

$$= \int\limits_0^1 \left(\frac{3x^2y^2}{2} + \frac{x^3}{3}\right)\Big|_{x=\sqrt{y}}^{x=1} dy = \int\limits_0^1 \left(\frac{3y^2}{2} + \frac{1}{3} - \frac{3y^3}{2} - \frac{y^{\frac{3}{2}}}{3}\right) dy$$

$$= \left(\frac{y^2}{2} + \frac{y}{3} - \frac{3y^4}{8} - \frac{2y^{\frac{5}{2}}}{15}\right)\Big|_0^1 = \frac{13}{40}. \quad \blacktriangle$$

3. Change of Variables in the Double Integral. Sometimes, for simplifying the computation of double integrals, it is convenient to

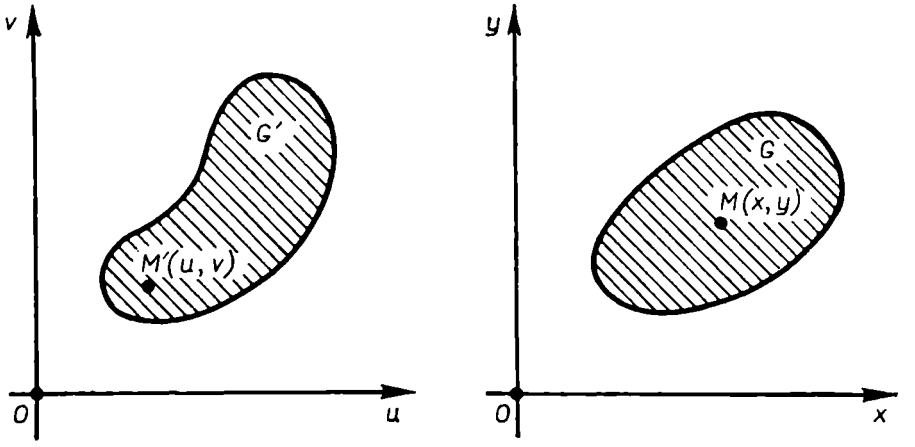

Fig. 143

use the formula for changing variables. Let us dwell on this topic in more detail.

Suppose it is required to compute the double integral of a continuous function $f(x, y)$, $(x, y) \in G$, where G is a bounded domain in the xy-plane.

Let the formulas

$$x = \varphi(u, v) \text{ and } y = \psi(u, v) \tag{1}$$

set one-to-one correspondence between the points $M(x, y) \in G$ and $M'(u, v)$ belonging to some domain G' in the uv-plane (see Fig. 143).

Thus, by formulas (1) the domain G' in the uv-plane is transformed into the domain G in the xy-plane, and by the formulas

$$u = \varphi_1(x, y) \text{ and } v = \psi_1(x, y), \tag{2}$$

which are obtained from formulas (1), if the latter are solved with respect to u and v, the domain G in the xy-plane is transformed into the domain G' in the uv-plane.

Since, given the coordinates of the point $M'(u, v)$, we can determine the position of the point $M(x, y) \in G$, being with M' in one-to-one correspondence, the coordinates u and v of the point M' can be regarded as some coordinates of the point M. The coordinates u and v are said to be curvilinear coordinates of the point M. Thus, formulas (1) and formulas (2) derived from them are formulas for transforming coordinates which enable us to find the curvilinear coordinates u and v of the point M given the rectangular coordinates of this point, and vice versa.

Let us give without proof the formula for change of variables in the double integral

$$\iint\limits_G f(x, y)\,dx\,dy = \iint\limits_G f(\varphi(u, v), \psi(u, v))\,|J|\,du\,dv, \qquad (3)$$

where

$$J(u, v) = \begin{vmatrix} \varphi'_u(u, v) & \psi'_u(u, v) \\ \varphi'_u(u, v) & \psi'_v(u, v) \end{vmatrix}$$

is called the Ostrogradskian (determinant) for system (1) or Jacobian for this system.

Note that the most widely used change of variables in computing double integrals is the replacement of the rectangular coordinates x and y by the polar coordinates r and φ by the known formulas $x = r\cos\varphi$ and $y = r\sin\varphi$. Since

$$J(r, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r,$$

bearing in mind that $|J| = r$, formula (3) for change of variables in the double integral takes the form:

$$\iint\limits_G f(x, y)\,dx\,dy = \iint\limits_G f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi. \qquad (4)$$

In order to compute a double integral, it should be replaced by an iterated integral. Let the domain of integration be the domain G:

$$\alpha \leqslant \varphi \leqslant \beta,\ r_1(\varphi) \leqslant r \leqslant r_2(\varphi),$$

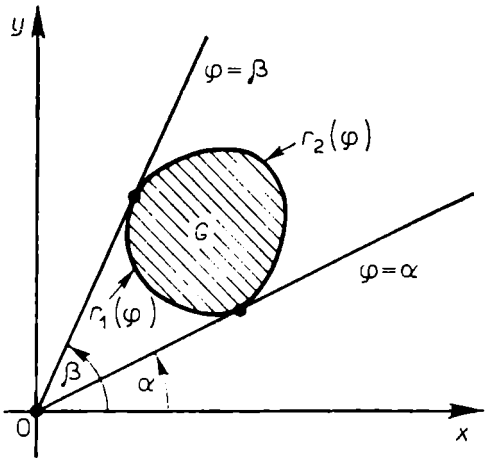

Fig. 144

where $r_1(\varphi)$ and $r_2(\varphi)$ are continuous functions on the interval $[\alpha, \beta]$ (Fig. 144). Then, by analogy with formulas (3) or (4) in the preceding subsection, we have the following formula:

$$\iint\limits_G f(r\cos\varphi,\ r\sin\varphi)\, r\, dr\, d\varphi = \int\limits_\alpha^\beta d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi,\ r\sin\varphi)\cdot r\, dr. \quad (5)$$

Example. Changing to polar coordinates, evaluate the double integral $\iint\limits_G \frac{dx\,dy}{\sqrt{x^2+y^2}}$, where G is the first quadrant of the circle of radius $R=1$ centred at the point $O(0, 0)$ (Fig. 145*a*).

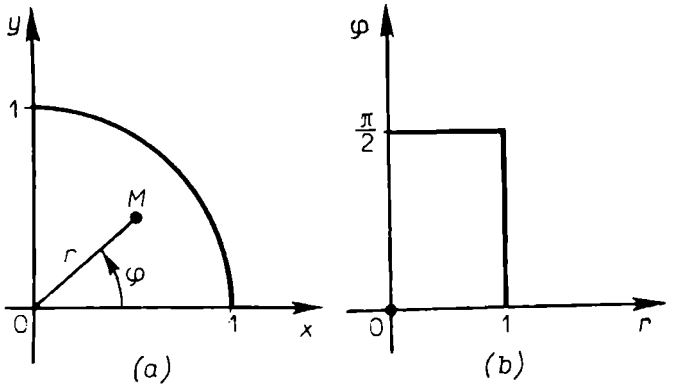

Fig. 145

△ Since $\sqrt{x^2+y^2}=r$, according to formula (4), we have:

$$\iint\limits_G \frac{dx\,dy}{\sqrt{x^2+y^2}} = \iint\limits_G \frac{r\,d\varphi\,dr}{r} = \iint\limits_G d\varphi\, dr.$$

The domain G in polar coordinates can be written with the help of the inequalities (see Fig. 145*b*)

$$0 \leqslant \varphi \leqslant \frac{\pi}{2}, \quad 0 \leqslant r \leqslant 1.$$

Using then formula (5), we get:

$$\iint\limits_G \frac{dx\,dy}{\sqrt{x^2+y^2}} = \iint\limits_G d\varphi\,dr = \int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^1 dr = \frac{\pi}{2}. \blacktriangle$$

Exercises

1. Evaluate the following iterated integrals:

(a) $\int\limits_3^5 dx \int\limits_0^1 xy\,dy$; (b) $\int\limits_2^4 dy \int\limits_0^1 xy^3\,dx$;

(c) $\int\limits_4^6 dy \int\limits_1^e \frac{y}{x}\,dx$; (d) $\int\limits_1^2 dx \int\limits_0^1 (x^3+y^3)\,dy$; (e) $\int\limits_1^3 dy \int\limits_2^4 \frac{x}{y^3}\,dx$.

2. Evaluate the given double integrals over the indicated rectangles:

(a) $\iint\limits_G x^3y\,dx\,dy$ if G: $2 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 2$;

(b) $\iint\limits_G \frac{x}{y^2}\,dx\,dy$ if G: $4 \leqslant x \leqslant 6$, $1 \leqslant y \leqslant 2$;

(c) $\iint\limits_G x \ln y\,dx\,dy$ if G: $0 \leqslant x \leqslant 4$, $1 \leqslant y \leqslant e$;

(d) $\iint\limits_G e^{x-y}\,dx\,dy$ if G: $-1 \leqslant x \leqslant 0$, $0 \leqslant y \leqslant 1$;

(e) $\iint\limits_G (x \sin y + y \cos x)\,dx\,dy$ if G: $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$;

(f) $\iint\limits_G \sin(x+y)\,dx\,dy$ if G: $0 \leqslant x \leqslant \frac{\pi}{2}$, $0 \leqslant y \leqslant \frac{\pi}{2}$.

3. Evaluate the given double integrals over the indicated domains:

(a) $\iint\limits_G (x+2y)\,dx\,dy$ if G: $2 \leqslant x \leqslant 3$, $x \leqslant y \leqslant 2x$;

(b) $\iint\limits_G (x+y)\,dx\,dy$ if G: $0 \leqslant x \leqslant 1$, $x \leqslant y \leqslant 2-x^2$.

4. Evaluate $\iint\limits_G y \ln x \, dx \, dy$ if the domain G is bounded by the lines $xy = 1$, $y = \sqrt{x}$, and $x = 2$.

5. Evaluate $\iint\limits_G x^2 y \, dx \, dy$ if the domain G is bounded by the lines $y = -x^2$ and $x = y^2$.

6. Evaluate $\iint\limits_G x \, dx \, dy$ if the domain G is a triangle with vertices $A(2, 3)$, $B(7, 2)$, and $C(4, 5)$.

7. Evaluate $\iint\limits_G (x^2 + y^2) \, dx \, dy$ if the domain G is a triangle with vertices $A(0, 0)$, $B(\sqrt{2}, \sqrt{2})$ and $C(\sqrt{2}, \sqrt{6})$.

8. Evaluate $\iint\limits_G r^3 \sin \varphi \, d\varphi \, dr$ if the domain G is a circular sector, bounded by the lines $\varphi = 0$, $\varphi = \frac{\pi}{2}$, and $r = 2$.

9. Changing to polar coordinates, evaluate $\iint\limits_G (x^2 + y^2) \, dx \, dy$ if the domain G is bounded by the lines $y = x$, $y = \sqrt{3x}$ and the arc of the circle $x^2 + y^2 = 8$ lying in the first quadrant.

10. Changing to polar coordinates, evaluate $\iint\limits_G \sqrt{x^2 + y^2} \, dx \, dy$, if the domain G is the first quadrant of the circle $x^2 + y^2 \leqslant 9$.

12.2. Double Integrals Applied to Physical and Geometrical Problems

1. Mass of a Plate. Suppose it is required to compute the mass distributed over a thin rectangular plate G (Fig. 146) with a variable surface density $\rho(x, y)$. The notion of surface density of a plate at a given point is similar to the considered notion of line density of a bar at a given point.

A plate is called homogeneous if any two of its parts of equal area have equal masses. As is known, if a plate is homogeneous, then its mass is found by the formula $m = \rho \cdot a \cdot b$, where ρ is surface density, and a and b are the length and width of the plate, respectively. A plate is said to be nonhomogeneous if two of its parts of equal area have, generally speaking, different masses. To compute the mass m of a nonhomogeneous plate, we proceed as follows. By straight lines $x = x_i = \frac{ia}{n}$ and $y = y_j = \frac{jb}{n}$ $(i, j = 0, 1, \ldots, n)$ which are para-

llel to the coordinate axes, we partition the rectangle G into n^2 rectangles G_{ij}, $i, j = 1, 2, \ldots, n$ (Fig. 147) and construct the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \rho(\tau_i, t_j) \Delta x_i \Delta y_i, \qquad (1)$$

where $\tau_i \in [x_{i-1}, x_i]$ and $t_j \in [y_{j-1}, y_j]$. The mass m of the plate G will be approximately expressed by sum (1). It is easy to see that the smaller the subintervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ $(i, j = 1, 2, \ldots, n)$, the better the approximation. Therefore for the mass m of the plate G we take the limit of sum (1) as $n \to \infty$. On the other hand, by definition, the limit of sum (1), as $n \to \infty$, is equal to the double integral of the function $\rho(x, y)$ over G. Therefore the mass m of the plate G with a variable surface density $\rho(x, y)$ is computed by the formula

$$m = \iint_G \rho(x, y)\, dx\, dy. \qquad (2)$$

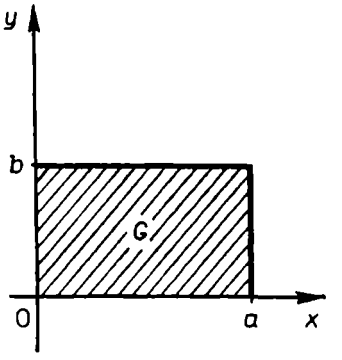

Fig. 146

In conclusion, we note that formula (2) remains valid for the case of a thin bounded plate of an arbitrary shape.

Example 1. Compute the mass m of a thin rectangular plate with sides a and b (see Fig. 146) if surface density $\rho(x, y) = x + y$.

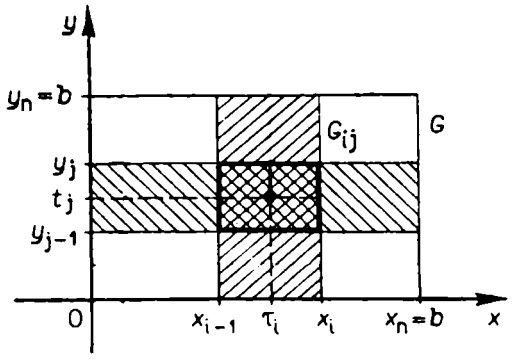

Fig. 147

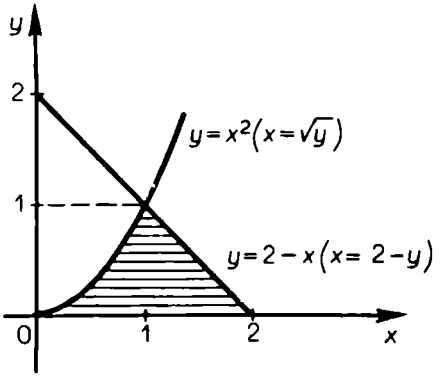

Fig. 148

△ To compute the mass m of the plate G, we make use of formula (2). We then have:

$$m = \iint_G (x + y)\, dx\, dy = \int_0^a dx \int_0^b (x + y)\, dy = \int_0^a \left(xy + \frac{y^2}{2} \right) \Big|_{y=0}^{y=b} dx$$

$$= \int_0^a \left(bx + \frac{b^2}{2} \right) dx = \left(b \frac{x^2}{2} + \frac{b^2}{2} x \right) \Big|_0^a = \frac{1}{2}(a^2 b + b^2 a) \quad ▲$$

Example 2. Compute the mass m of a thin plate G which is a curvilinear triangle bounded by the x-axis, parabola $y = x^2$, and straight line $x + y = 2$ (Fig. 148) if the surface density $\rho(x, y) = xy^2$.

△ To compute the mass m of the plate G: $\sqrt{y} \leqslant x \leqslant 2 - y$, $0 \leqslant y \leqslant 1$, we apply formula (2). We then have:

$$m = \iint_G xy^2\,dx\,dy = \int_0^1 dy \int_{\sqrt{y}}^{2-y} xy^2\,dx = \int_0^1 \frac{x^2y^2}{2}\Big|_{x=\sqrt{y}}^{x=2-y} dy$$

$$= \int_0^1 \left(\frac{(2-y)^2 y^2}{2} - \frac{y^3}{2}\right) dy = \int_0^1 \left(2y^2 - \frac{5}{2}y^3 + \frac{y^4}{2}\right) dy$$

$$= \left(\frac{2}{3}y^3 - \frac{5}{8}y^4 + \frac{y^5}{10}\right)\Big|_0^1 = \frac{17}{120}. \ ▲$$

2. Static Moments and the Centre of Gravity of a Plate. Consider now the problem on computing static moments of a thin rectangular plate G with respect to the coordinate axes (see Fig. 146). For this purpose, we take advantage of the partition of the plate G into n^2 rectangles G_{ij} each of which represents a rectangle with sides $(x_i - x_{i-1})$ and $(y_j - y_{j-1})$ (see Fig. 147). We shall assume that the mass of each of the rectangles G_{ij} $(i, j = 1, 2, \ldots, n)$ is concentrated at the point (τ_i, t_j). Then, using the formulas for finding static moments of a system of n material points studied in the course of physics, we shall have:

$$M_x^{(n)} = \sum_{i=1}^{n} \sum_{j=1}^{n} t_j \rho(\tau_i, t_j)\,\Delta x_i\,\Delta y_j, \tag{1}$$

$$M_y^{(n)} = \sum_{i=1}^{n} \sum_{j=1}^{n} \tau_i \rho(\tau_i, t_j)\,\Delta x_i\,\Delta y_j. \tag{2}$$

Passing to the limit, as $n \to \infty$, in relations (1) and (2) and taking into consideration the definition of the double integral, we get the formulas for finding static moments of the plate G with respect to the coordinate axes:

$$M_x = \iint_G y\rho(x, y)\,dx\,dy \quad \text{and} \quad M_y = \iint_G x\rho(x, y)\,dx\,dy. \tag{3}$$

It is known from mechanics that the static moment of a plate with respect to some axis coincides with static moment of a point mass equal to the mass of the plate concentrated at its centre of gravity with respect to the same axis. Therefore, denoting the coordinates of the centre of gravity (the centre of masses) of the plate G

in the coordinate system xOy by (x_0, y_0), we shall have:

$$M_x = my_0 \quad \text{and} \quad M_y = mx_0.$$

Hence, bearing in mind formula (3), we get:

$$x_0 = \frac{M_y}{m} = \frac{\iint\limits_G x\rho(x, y)\,dx\,dy}{\iint\limits_G \rho(x, y)\,dx\,dy}, \qquad y_0 = \frac{M_x}{m} = \frac{\iint\limits_G y\rho(x, y)\,dx\,dy}{\iint\limits_G \rho(x, y)\,dx\,dy}. \tag{4}$$

If the plate is homogeneous, that is, $\rho(x, y) = \text{const}$, then formulas (4) take the form:

$$x_0 = \frac{\iint\limits_G x\,dx\,dy}{S} \quad \text{and} \quad y_0 = \frac{\iint\limits_G y\,dx\,dy}{S}, \tag{5}$$

where S is the area of the plate G.

In closing, we should like to note that formulas (4) and (5) remain valid for the case of a thin bounded plate of an arbitrary shape.

Example. Find the coordinates of the centre of gravity of a rectangular plate G: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$ if the surface density $\rho(x, y) = x + y$.

△ Using formula (2) of the preceding subsection, we find the mass of the plate:

$$m = \iint\limits_G (x+y)\,dx\,dy = \int\limits_0^1 dx \int\limits_0^2 (x+y)\,dy$$

$$= \int\limits_0^1 \left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{y=2} dx = \int\limits_0^1 (2x+2)\,dx = (x^2+2x)\Big|_0^1 = 3.$$

Using formulas (3), we compute the static moments of the plate G with respect to the coordinate axes:

$$M_x = \iint\limits_G y(x+y)\,dx\,dy = \int\limits_0^1 dx \int\limits_0^2 (xy+y^2)\,dy$$

$$= \int\limits_0^1 \left(\frac{xy^2}{2} + \frac{y^3}{3}\right)\Big|_{y=0}^{y=2} dx = \int\limits_0^1 \left(2x + \frac{8}{3}\right) dx = \left(x^2 + \frac{8}{3}x\right)\Big|_0^1 = \frac{11}{3};$$

$$M_y = \iint\limits_G x(x+y)\,dx\,dy = \int\limits_0^1 dx \int\limits_0^2 (x^2+xy)\,dy$$

$$= \int_0^1 \left(x^2y + \frac{xy^2}{2}\right)\Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 (2x^2 + 2x)\, dx = \left(\frac{2}{3}x^3 + x^2\right)\Big|_0^1 = \frac{5}{3}.$$

On the basis of formulas (4), the centre of gravity of the plate G has the following coordinates:

$$x_0 = \frac{M_y}{m} = \frac{5}{3 \cdot 3} = \frac{5}{9} \quad \text{and} \quad y_0 = \frac{M_x}{m} = \frac{11}{3 \cdot 3} = \frac{11}{9}. \blacktriangle$$

3. Moments of Inertia of a Plate. It is known from mechanics that the moment of inertia of a material point M of mass m with respect to some axis is defined as the product of the mass by the square of the distance from the point M to this axis. The moment of inertia with respect to the axis of a system of material points is equal to the sum of moments of inertia of all points with respect to the same axis. Therefore for finding the moments of inertia with respect to the coordinate axes of a thin rectangular plate G on which the mass with the surface density $\rho(x, y)$ is distributed, we can use the same method which was utilized above for computing the static moments of the rectangular plate G with respect to the coordinate axes. Thus, we shall have:

$$I_x = \iint_G y^2 \rho(x, y)\, dx\, dy \quad \text{and} \quad I_y = \iint_G x^2 \rho(x, y)\, dx\, dy. \tag{1}$$

The moment of inertia I_l of the plate G with respect to some arbitrary axis l is computed by the formula

$$I_l = \iint_G r^2 \rho(x, y)\, dx\, dy, \tag{2}$$

where r is the distance from the point $M(x, y)$ of the plate G to the axis l.

In particular, the moment of inertia of the plate G with respect to the origin, that is, with respect to the axis passing through origin perpendicular to the xy-plane will be equal to

$$I_0 = \iint_G (x^2 + y^2)\, \rho(x, y)\, dx\, dy, \tag{3}$$

since in this case $r^2 = x^2 + y^2$. Then

$$I_0 = I_x + I_y.$$

Note that the derived formulas are also valid for the case of a thin bounded plate of an arbitrary shape.

Example. Compute the moment of inertia of a homogeneous plate G: $-1 \leqslant x \leqslant 1$, $x^2 \leqslant y \leqslant 1$ with respect to the axis $y = -1$ (Fig. 149).

△ Since, by hypothesis, the plate is homogeneous, its density $\rho(x, y) = \text{const}$. To simplify the computations, we set $\rho(x, y) = 1$. The distance r from any point $M(x, y)$ of the plate G to the axis $y = -1$ is computed by the formula $r = y + 1$. Therefore, using formula (2) and bearing in mind that the figure G is symmetric about the y-axis, we have

$$I_l = \iint\limits_G (y+1)^2\,dx\,dy = 2\iint\limits_{G_0} (y+1)^2\,dx\,dy$$

$$= 2\int\limits_0^1 (y+1)^2\,dy \int\limits_0^{\sqrt{y}} dx = 2\int\limits_0^1 (y+1)^2\,x\Big|_{x=0}^{x=\sqrt{y}}\,dy$$

$$= 2\int\limits_0^1 (y+1)^2\sqrt{y}\,dy = 2\int\limits_0^1 \left(y^{\frac{5}{2}} + 2y^{\frac{3}{2}} + y^{\frac{1}{2}}\right) dy$$

$$= 2\left(\frac{2}{7}\,y^{\frac{7}{2}} + \frac{4}{5}\,y^{\frac{5}{2}} + \frac{2}{3}\,y^{\frac{3}{2}}\right)\Big|_0^1 = \frac{368}{105}\,. \quad ▲$$

4. The Volume of a Solid. Let a nonnegative continuous function $z = f(x, y)$ be given on a rectangle G: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. Let us find the formula for computing the volume V of the body the coordinates of points of which satisfy the conditions $(x, y) \in G$, $0 \leqslant z \leqslant f(x, y)$. Such solid is called a cylindrical solid.

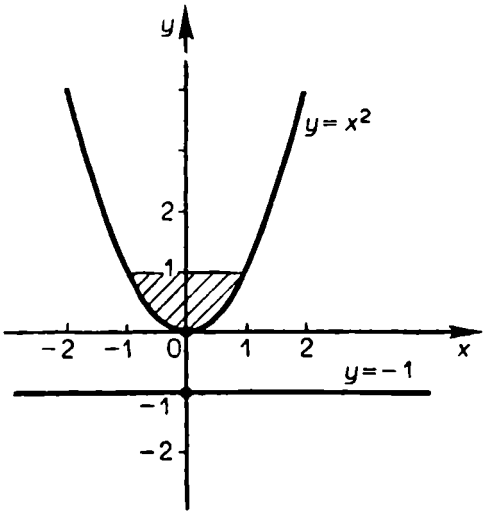

Fig. 149

For the continuous function $f(x, y)$, $(x, y) \in G$, we form the integral sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(\tau_i, t_j)\,\Delta x_i\,\Delta y_j.$$

This sum, approximately equal to the volume V, obviously, tends to V as $n \to \infty$. Thus, taking into account the definition of the double integral, we have

$$V = \iint\limits_G f(x, y)\,dx\,dy. \tag{1}$$

Note that formula (1) is also valid for the case when G is an arbitrary bounded domain. Formula (1) is used for computing the volume of cylindrical solids.

If in formula (1) we set $f(x, y) = 1$, then we shall get the volume of a cylindrical solid with altitude $z = 1$ numerically equal to the

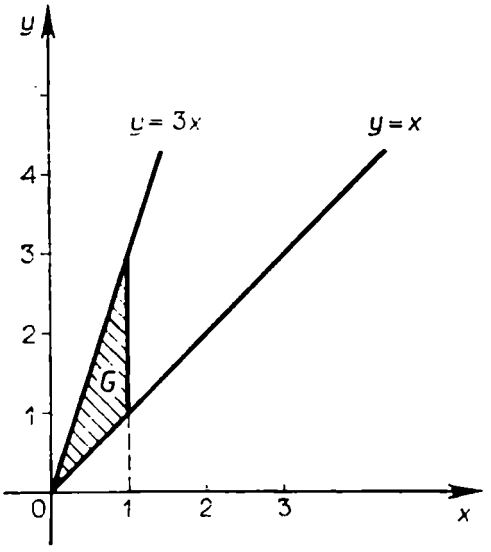

Fig. 150

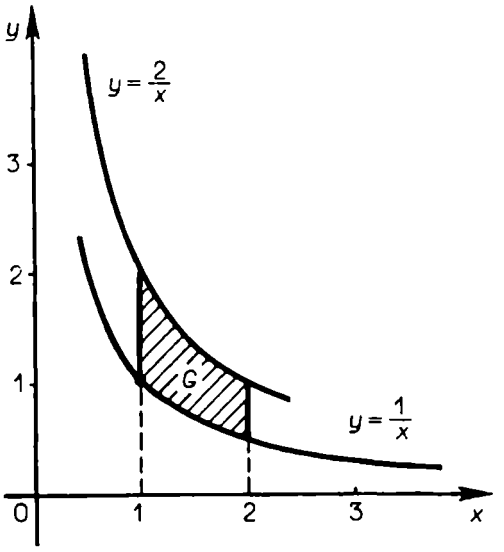

Fig. 151

area S of its base G. Therefore the area S of a plane domain G can be computed by the formula

$$S = \iint\limits_G dx\, dy. \tag{2}$$

Example 1. Find the volume of a cylindrical solid bounded by the planes $x = 1$, $y = x$, $y = 3x$, $z = 0$, and the surface $z = x^2 + y^2$.

△ We represent in Fig. 150 the domain G: $0 \leqslant x \leqslant 1$, $x \leqslant y \leqslant 3x$. To compute the volume V, we make use of formula (1):

$$V = \iint\limits_G (x^2 + y^2)\, dx\, dy = \int\limits_0^1 dx \int\limits_x^{3x} (x^2 + y^2)\, dy = \int\limits_0^1 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=x}^{y=3x} dx$$

$$= \int\limits_0^1 \left(\frac{32x^3}{3} \right) dx = \frac{8}{3} x^4 \Big|_0^1 = \frac{8}{3}. \ ▲$$

Example 2. Find the area S of a plane domain G given by the inequalities $1 \leqslant x \leqslant 2$, $\frac{1}{x} \leqslant y \leqslant \frac{2}{x}$ (Fig. 151).

△ For computing the area of the domain G, we make use of formula (2). Then, we get

$$S = \iint\limits_G dx\,dy = \int\limits_1^2 dx \int\limits_{\frac{1}{x}}^{\frac{2}{x}} dy = \int\limits_1^2 \left(\frac{2}{x} - \frac{1}{x}\right) dx$$

$$= \int\limits_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2. \blacktriangle$$

Exercises

1. Compute the area of the domain bounded by the lines

$$x = y^2 + 1, \quad x = 5.$$

2. Compute the area of the domain bounded by the lines

$$x + y = 6, \quad x = 4y - y^2.$$

3. Compute the area of the domain bounded by the lines

$$y = 2 - x^2, \quad y^3 = x^2.$$

4. Compute the mass of a square plate with side b whose surface density at any point is proportional to the square of the distance from this point to one of the vertices of the square.

5. Compute the mass of a square plate with side b whose surface density at any point is equal to $k(b^2 - r^2)$, where r is the distance from this point to the centre of the square and k the proportionality factor.

6. Compute the mass of a triangular plate G: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant x$ whose surface density $\rho = 1 + xy$.

7. Compute the coordinates of the centre of gravity of the homogeneous plates bounded by the lines:

(a) $x = -a$, $x = a$, $y = -b$, $y = b$;

(b) $x = 0$, $x = a$, $y = 0$, $y = b$;

(c) $y = x$, $y = 4 - x$, $x = 4 + 2y$;

(d) $y = x^2$, $x + y = 2$;

(e) $y^2 = 4x + 4$, $y^2 = -2x + 4$.

8. Compute the static moments and coordinates of the centre of gravity of the domain lying in the first quadrant and bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ and coordinate axes if the surface density at each point of the domain is proportional to the product of the coordinates of this point.

9. Compute the static moment of the plate, having the shape of a right triangle OAB with legs $OA = a$, $OB = b$, with respect to the leg OA if the surface density at each point of the domain is numerically equal to the distance from this point to the leg OA.

10. Compute the static moments of a semicircular plate G: $-R \leqslant x \leqslant R$, $0 \leqslant y \leqslant \sqrt{R^2 - x^2}$, with respect to the coordinate axes.

11. Compute the moment of inertia of a homogeneous plate bounded by the lines $\frac{x}{3} + \frac{y}{2} = 1$, $x = 0$, and $y = 0$, with respect to the origin of coordinates.

12. Compute the moment of inertia of a homogeneous plate having the shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to its major axis.

13. Compute the volume of the solid bounded by the surfaces $y = 1 + x^2$, $z = 3x$, $y = 5$ and situated in the first octant.

14. Compute the volume of the solid bounded by the surfaces $x^2 + y^2 = 8$, $x = 0$, $y = 0$, $z = 0$, $x + y + z = 4$.

15. Compute the volume of the body bounded by the surfaces $x^2 + y^2 = 4$, $x^2 + z^2 = 4$.

12.3. Triple Integrals

1. The Definition of the Triple Integral. Let us first give the definition of the triple integral of a function $f(x, y, z)$ defined on a rectangular parallelepiped D: $a_1 \leqslant x \leqslant b_1$, $a_2 \leqslant y \leqslant b_2$, $a_3 \leqslant z \leqslant b_3$.

For this purpose, as in defining the double integral in the case of a rectangle, we divide each of the intervals $[a_1, b_1]$, $[a_2, b_2]$, and $[a_3, b_3]$ on the coordinate axes by the points

$$x_i = a_1 + \frac{b_1 - a_1}{n} i \quad (i = 0, 1, 2, \ldots, n),$$

$$y_j = a_2 + \frac{b_2 - a_2}{n} j \quad (j = 0, 1, 2, \ldots, n),$$

$$z_k = a_3 + \frac{b_3 - a_3}{n} k \quad (k = 0, 1, 2, \ldots, n)$$

into n equal subintervals and form the integral sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\tau_i, t_j, \mu_k) \Delta x_i \Delta y_j \Delta z_k,$$

where $\tau_i \in [x_{i-1}, x_i]$, $t_j \in [y_{j-1}, y_j]$, and $\mu_k \in [z_{k-1}, z_k]$. If the integral sum has a limit, as $n \to \infty$, which is independent of the choice of τ_i, t_j, and μ_k, then it is called the triple integral over the paralle-

lepiped D and is denoted by

$$\iiint\limits_D f(x,\ y,\ z)\,dx\,dy\,dz.$$

For this integral, the same as for the double integral, the following formulas are established:

$$\iiint\limits_D f(x,\ y,\ z)\,dx\,dy\,dz = \int\limits_{a_1}^{b_1} dx \int\limits_{a_2}^{b_2} dy \int\limits_{a_3}^{b_3} f(x,\ y,\ z)\,dz,$$

$$\iiint\limits_D f(x,\ y,\ z)\,dx\,dy\,dz = \int\limits_{a_2}^{b_2} dy \int\limits_{a_1}^{b_1} dx \int\limits_{a_3}^{b_3} f(x,\ y,\ z)\,dz,$$

$$\iiint\limits_D f(x,\ y,\ z)\,dx\,dy\,dz = \int\limits_{a_3}^{b_3} dz \int\limits_{a_2}^{b_2} dy \int\limits_{a_1}^{b_1} f(x,\ y,\ z)\,dx$$

and so on. In all, as it is easy to see, there are six formulas for reducing the triple integral to iterated integrals.

Example. Evaluate the triple integral of the function $f(x, y, z) = x^2 + xz + xyz$ over the parallelepiped D: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$.

△ To calculate the triple integral, we make use of one of the formulas for reducing the triple integral to an iterated integral. Then

$$\begin{aligned}
\iiint\limits_D f(x,\ y,\ z)\,dx\,dy\,dz &= \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 (x^2 + xz + xyz)\,dz \\
&= \int\limits_0^1 dx \int\limits_0^1 \left(x^2 z + \frac{xz^2}{2} + \frac{xyz^2}{2}\right)\Big|_{z=0}^{z=1} dy \\
&= \int\limits_0^1 dx \int\limits_0^1 \left(x^2 + \frac{x}{2} + \frac{xy}{2}\right) dy \\
&= \int\limits_0^1 \left(x^2 y + \frac{xy}{2} + \frac{xy^2}{4}\right)\Big|_{y=0}^{y=1} dx \\
&= \int\limits_0^1 \left(x^2 + \frac{x}{2} + \frac{x}{4}\right) dx \\
&= \left(\frac{x^3}{3} + \frac{3x^2}{8}\right)\Big|_0^1 = \frac{17}{24}. \ \blacktriangle
\end{aligned}$$

Note that it is possible to introduce the definition of the triple integral for a function given in a bounded three-dimensional domain of an arbitrary shape, as it was done earlier for the double integral.

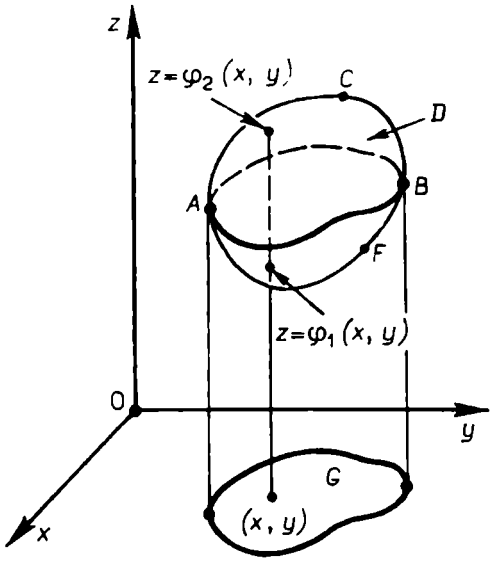

Fig. 152

The properties of the triple integral are analogous to those of the double integral: the integral of the sum (difference) of functions is equal to the sum (difference) of the integrals; a constant factor can be taken outside the integral sign.

Consider the bounded closed domain D of an arbitrary shape in which the function $f(x, y, z)$ is integrable (Fig. 152). Let G denote the projection of the domain D on the xy-plane. Suppose that the straight lines parallel to the z-axis intersect the boundary of the domain D at no more than two points. Then on the surface bounding the domain D we can single out the lower and upper parts (AFB and ACB, respectively). Let $z=\varphi_1(x, y)$ be the equation of the lower part, and $z=\varphi_2(x, y)$ the equation of the upper part. If for any point $(x, y) \in G$ the integral

$$\int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z)\, dz$$

exists, then it is possible to prove that

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iint_G dx\, dy \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z)\, dz.$$

There are another two formulas:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iint_{G_1} dx\, dz \int_{\psi_1(x, z)}^{\psi_2(x, z)} f(x, y, z)\, dy,$$

where G_1 is the projection of D on the xz-plane, and $y=\psi_1(x, z)$ and $y=\psi_2(x, z)$ are the equations of the left-hand and right-hand parts of the surface bounding the domain D, respectively:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iint_{G_2} dy\, dz \int_{\chi_1(y, z)}^{\chi_2(y, z)} f(x, y, z)\, dx,$$

where G_2 is the projection of D on the yz-plane, $x = \chi_1(y, z)$ and $x = \chi_2(y, z)$ are respectively the equations of the front and rear parts of the surface bounding the domain D.

Since the double integral is reduced to computing an iterated integral, consequently, the computation of the triple integral is reduced to three successive simple integrations.

2. Applications of the Triple Integral in Physics and Geometry. Similarly to the formula for computing the mass of a thin plate with a variable surface density, we can prove the validity of the following formula for computing the mass of the solid D with a variable spatial density $\rho = \rho(x, y, z)$, $(x, y, z) \in D$:

$$m = \iiint_D \rho(x, y, z)\, dx\, dy\, dz. \tag{1}$$

Example 1. Compute the mass m of the cube D: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$, if the density $\rho(x, y, z) = 1 + xyz$.

△ By formula (1), we have:

$$m = \iiint_D (1 + xyz)\, dx\, dy\, dz = \int_0^1 dx \int_0^1 dy \int_0^1 (1 + xyz)\, dz$$

$$= \int_0^1 dx \int_0^1 \left(z + \frac{xyz^2}{2}\right)\Big|_{z=0}^{z=1} dy = \int_0^1 dx \int_0^1 \left(1 + \frac{xy}{2}\right) dy$$

$$= \int_0^1 \left(y + \frac{xy^2}{4}\right)\Big|_{y=0}^{y=1} dx = \int_0^1 \left(1 + \frac{x}{4}\right) dx = \left(x + \frac{x^2}{8}\right)\Big|_0^1 = \frac{9}{8}. \ ▲$$

For computing the moment of inertia (about some axis) of a solid whose mass is distributed over a spatial bounded domain D with a variable density $\rho = \rho(x, y, z)$, $(x, y, z) \in D$, we can derive the following formula:

$$J_l = \iiint_D r^2(x, y, z)\, \rho(x, y, z)\, dx\, dy\, dz, \tag{2}$$

where $r(x, y, z)$ is the distance from points $M(x, y, z)$ of the solid D to the axis l.

If the axis l coincides with the z-axis, then $r^2 = x^2 + y^2$, and formula (2) takes the form:

$$J_z = \iiint_D (x^2 + y^2)\, \rho(x, y, z)\, dx\, dy\, dz. \tag{3}$$

If the axis l coincides with the x-axis, then $r^2 = y^2 + z^2$, and formula (2) takes the form:

$$J_x = \iiint\limits_D (y^2+z^2)\,\rho(x,\ y,\ z)\,dx\,dy\,dz. \tag{4}$$

And if the axis l coincides with the y-axis, then $r^2 = x^2 + z^2$, and formula (2) can be written in the form

$$J_y = \iiint\limits_D (x^2+z^2)\,\rho(x,\ y,\ z)\,dx\,dy\,dz. \tag{5}$$

Example 2. Find the moment of inertia of a homogeneous cube D: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$ about its edge.

△ Since we are given a homogeneous cube, its density $\rho(x, y, z) = \text{const}$. To make the computations simple, we set $\rho(x, y, z) = 1$.

To find the moment of inertia of the cube about its edge, we may use any of formulas (3), (4), and (5) (since the edges of the cube emanating from one and the same vertex are mutually orthogonal, the coordinate system $Oxyz$ can be chosen so that they will lie on the coordinate axes):

$$J_x = \iiint\limits_D (y^2+z^2)\,dx\,dy\,dz = \int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^1 (y^2+z^2)\,dz$$

$$= \int\limits_0^1 dx \int\limits_0^1 \left(y^2 z + \frac{z^3}{3}\right)\Big|_{z=0}^{z=1} dy = \int\limits_0^1 dx \int\limits_0^1 \left(y^2 + \frac{1}{3}\right) dy$$

$$= \int\limits_0^1 \left(\frac{y^3}{3} + \frac{y}{3}\right)\Big|_{y=0}^{y=1} dx = \int\limits_0^1 \frac{2}{3}\,dx = \frac{2}{3}. \ ▲$$

For finding the static moments of the solid D with a variable density $\rho = \rho(x, y, z)$, $(x, y, z) \in D$, with respect to the coordinate planes, the following formulas are used:

$$M_{yz} = \iiint\limits_D x\rho(x,\ y,\ z)\,dx\,dy\,dz,$$

$$M_{zx} = \iiint\limits_D y\rho(x,\ y,\ z)\,dx\,dy\,dz, \tag{6}$$

$$M_{xy} = \iiint\limits_D z\rho(x,\ y,\ z)\,dx\,dy\,dz.$$

If we denote the coordinates of the centre of gravity (the centre of masses) of the solid D in the $Oxyz$ coordinate system by (x_0, y_0, z_0), then we get the following formulas for their computation:

$$x_0 = \frac{M_{yz}}{m}, \quad y_0 = \frac{M_{zx}}{m}, \quad z_0 = \frac{M_{xy}}{m}.$$

Taking into account formulas (1) and (6), we get:

$$x_0 = \frac{\iiint\limits_D x\rho(x, y, z)\,dx\,dy\,dz}{\iiint\limits_D \rho(x, y, z)\,dx\,dy\,dz}, \quad y_0 = \frac{\iiint\limits_D y\rho(x, y, z)\,dx\,dy\,dz}{\iiint\limits_D \rho(x, y, z)\,dx\,dy\,dz},$$

$$z_0 = \frac{\iiint\limits_D z\rho(x, y, z)\,dx\,dy\,dz}{\iiint\limits_D \rho(x, y, z)\,dx\,dy\,dz}. \tag{7}$$

Example 3. Find the coordinates of the centre of gravity (the centre of masses) of the homogeneous solid D: $0 \leqslant x \leqslant 3$, $1 \leqslant y \leqslant 3$, $0 \leqslant z \leqslant \frac{1}{2}(3 - x)$ (Fig. 153).

△ Since, by hypothesis, we are given a homogeneous solid, its density $\rho(x, y, z) = \text{const}$. To simplify the calculations, we shall

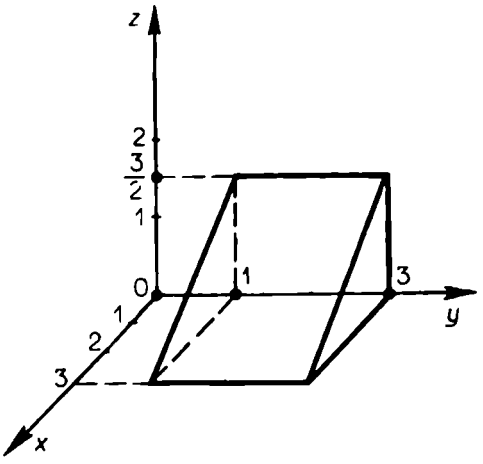

Fig. 153

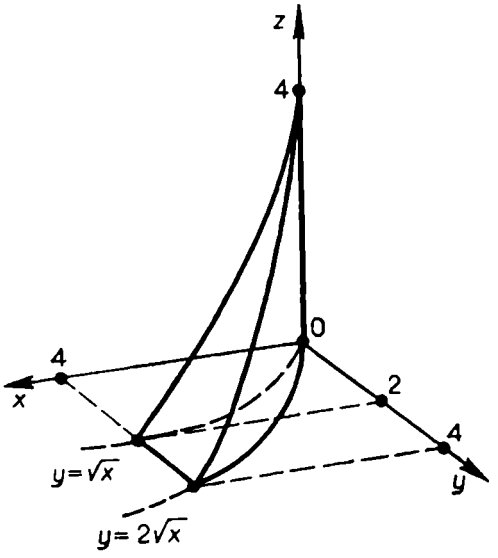

Fig. 154

assume that $\rho(x, y, z) = 1$. We find the mass of the solid by formula (1):

$$m = \iiint\limits_D dx\,dy\,dz = \int\limits_0^3 dx \int\limits_1^3 dy \int\limits_0^{\frac{1}{2}(3-x)} dz = \int\limits_0^3 dx \int\limits_1^3 z\Big|_{z=0}^{z=\frac{1}{2}(3-x)} dy$$

$$= \frac{1}{2}\int\limits_0^3 dx \int\limits_1^3 (3-x)\,dy = \frac{1}{2}\int\limits_0^3 (3-x)\,y\Big|_{y=1}^{y=3} dx$$

$$= \int\limits_0^3 (3-x)\,dx = \left(3x - \frac{x^2}{2}\right)\Big|_0^3 = \frac{9}{2}.$$

We find the static moments by formulas (6):

$$M_{yz}=\iiint\limits_D x\,dx\,dy\,dz=\int\limits_0^3 x\,dx\int\limits_1^3 dy\int\limits_0^{\frac{1}{2}(3-x)} dz$$

$$=\int\limits_0^3 x\,dx\int\limits_1^3 \frac{3-x}{2}\,dy=\int\limits_0^3 x(3-x)\,dx=\left(\frac{3x^2}{2}-\frac{x^3}{3}\right)\Big|_0^3=\frac{9}{2};$$

$$M_{zx}=\iiint\limits_D y\,dx\,dy\,dz=\int\limits_0^3 dx\int\limits_1^3 y\,dy\int\limits_0^{\frac{1}{2}(3-x)} dz$$

$$=\int\limits_0^3 dx\int\limits_1^3 y\,\frac{(3-x)}{2}\,dy=\int\limits_0^3 \frac{(3-x)}{2}\,\frac{y^2}{2}\Big|_{y=1}^{y=3}dx$$

$$=\int\limits_0^3 2(3-x)\,dx=2\left(3x-\frac{x^2}{2}\right)\Big|_0^3=9;$$

$$M_{xy}=\iiint\limits_D z\,dx\,dy\,dz=\int\limits_0^3 dx\int\limits_1^3 dy\int\limits_0^{\frac{1}{2}(3-x)} z\,dz$$

$$=\int\limits_0^3 dx\int\limits_1^3 \frac{z^2}{2}\Big|_{z=0}^{z=\frac{3-x}{2}}dy=\int\limits_0^3 \frac{(3-x)^2}{8}\,dx\int\limits_1^3 dy$$

$$=\frac{1}{4}\int\limits_0^3 (3-x)^2\,dx=\frac{1}{4}\left(-\frac{(3-x)^3}{3}\right)\Big|_0^3=\frac{9}{4}.$$

Now we find the coordinates of the centre of gravity:

$$x_0=\frac{M_{yz}}{m}=1,\quad y_0=\frac{M_{zx}}{m}=2,\quad \text{and}\quad z_0=\frac{M_{xy}}{m}=\frac{1}{2}.\ \blacktriangle$$

Since the mass of the solid is computed by the formula

$$m=\iiint\limits_D \rho(x,\,y,\,z)\,dx\,dy\,dz,$$

in the case $\rho(x,\,y,\,z)=1$ the mass of the domain (solid) D will be numerically equal to its volume. Therefore the volume of the solid will be numerically expressed by the triple integral

$$V=\iiint\limits_D dx\,dy\,dz.$$

Example 4. Compute the volume of the solid D: $0\leqslant x\leqslant 4$, $\sqrt{x}\leqslant y\leqslant 2\sqrt{x}$, $0\leqslant z\leqslant 4-x$ (Fig. 154).

△ To compute the volume of the solid D, we apply formula (8):

$$V=\iiint_D dx\,dy\,dz=\int_0^4 dx\int_{\sqrt{x}}^{2\sqrt{x}} dy\int_0^{4-x} dz=\int_0^4 dx\int_{\sqrt{x}}^{2\sqrt{x}} z\Big|_{z=0}^{z=4-x} dy$$

$$=\int_0^4 dx\int_{\sqrt{x}}^{2\sqrt{x}}(4-x)\,dy=\int_0^4 (4-x)\,y\Big|_{y=\sqrt{x}}^{y=2\sqrt{x}} dx$$

$$=\int_0^4 (4-x)(2\sqrt{x}-\sqrt{x})\,dx=\int_0^4 (4-x)\sqrt{x}\,dx$$

$$=\left(8\frac{x^{\frac{3}{2}}}{3}-\frac{2}{5}x^{\frac{5}{2}}\right)\Big|_0^4=\frac{128}{15}.\ \blacktriangle$$

Exercises

1. Evaluate the following triple integrals:

(a) $\int_0^1 dx\int_0^3 dy\int_0^2 (x+y+z)$; (b) $\int_0^1 dx\int_0^1 dy\int_0^1 (3x+y+2z)\,dz$;

(c) $\int_0^2 dx\int_0^1 dy\int_0^{x+2y} dz$; (d) $\int_0^1 dx\int_0^x dy\int_0^{\sqrt{x^2+y^2}} z\,dz$.

2. Evaluate the triple integrals over the indicated domains:

(a) $\iiint_D (x+2y+3z+4)\,dx\,dy\,dz$ if D: $0\leqslant x\leqslant 3$, $0\leqslant y\leqslant 2$;

(b) $\iiint_D (4x+3y+2z+1)\,dx\,dy\,dz$ if D: $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 2$, $0\leqslant z\leqslant 3$;

(c) $\iiint_D (x+xy+xyz)\,dx\,dy\,dz$ if D: $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$, $0\leqslant z\leqslant 1$.

3. Evaluate $\iiint_D x\,dx\,dy\,dz$ if the domain D is bounded by the planes $x=0$, $y=0$, $z=0$, $y=1$, and $x+z=2$.

4. Evaluate $\iiint\limits_D (3x+4y)\,dx\,dy\,dz$ if the domain D is bounded by the surfaces

$$z=x^2+y^2,\quad y=x^2,\quad z=0,\quad y=1.$$

5. Evaluate $\iiint\limits_D z\,dx\,dy\,dz$ if D: $0\leqslant x\leqslant\frac{1}{2}$, $x\leqslant y\leqslant 2x$, $0\leqslant z\leqslant\sqrt{1-x^2-y^2}$.

6. Compute the volume of the solid bounded by the following surfaces:

(a) $2x+3y+4z=12$, $x=0$, $y=0$, $z=0$;

(b) $3z=x^2+y^2$, $z=3$;

(c) $x^2+y^2+z^2-3y=0$, $x^2+z^2=2-y$.

7. Compute the mass of the cube $0\leqslant x\leqslant 3$, $0\leqslant y\leqslant 3$, $0\leqslant z\leqslant 3$ if the solid density $\rho(x, y, z)=x+y+z$.

8. Compute the mass of the solid bounded by the surfaces $z=xy$, $y=x$, $x=1$ if the volume density $\rho(x, y, z)=xy^2z^3$.

9. Compute the mass of the solid bounded by the paraboloid of revolution $x^2+y^2=2z$ and the plane $z=2$ if the volume density is proportional to the sum of squares of first two coordinates of a given point $M(x, y, z)$.

10. Compute the coordinates of the centre of gravity of homogeneous solids bounded by the following surfaces:

(a) $x=0$, $y=0$, $z=1$, $z=3$, and $2x+y=3$;

(b) $x+y+z=4$, $x=0$, $y=0$, and $z=0$;

(c) $x^2+2y^2=4z$, $z=2$;

(d) $x^2+y^2+z^2=1$, $x^2+y^2+z^2=2y$.

11. Compute the moments of inertia about the z-axis of homogeneous solids bounded by the surfaces:

(a) $x+y+z=1$, $x=0$, $y=0$, and $z=0$;

(b) $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$.

12. Compute the moment of inertia about the y-axis of the homogeneous solid bounded by the surfaces

$$x^2+y^2+z^2=2,\quad x^2+z^2=y^2\quad (y>0).$$

13. Compute the moment of inertia about the z-axis of the homogeneous solid bounded by the surfaces

$$x^2+y^2=2,\quad z=0.$$

12.4. Basic Concepts of Vector Analysis

1. Scalar and Vector Fields. Vectors Dependent on a Scalar Argument. Physics studies quantities of two kinds: scalars and vectors. A scalar is a physical quantity which requires only a number and a unit for its specification. Examples of scalars are mass, density, energy, etc. A vector is a quantity which involves direction and magnitude, and must have both for a full description. Examples of vector quantities are velocity, acceleration, force, a directed line segment in space, etc.

Scalar and vector quantities can be functions of the coordinates of points in space and time. In this case, they are called scalar and vector fields, respectively.

The following are examples of scalar fields: a temperature field inside a heated body, an illumination field generated by some light source, etc. A velocity field of a fluid flux, an electrostatic field generated by electric charges distributed in space, etc. are examples of vector fields.

In mechanics, time is most often the scalar argument of vector quantities; it is denoted by t.

Suppose we are given a vector function $\mathbf{a} = \mathbf{a}(t)$. If we resolve the vector $\mathbf{a}(t)$ into its components in the directions of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of a rectangular Cartesian coordinate system, then the vector function will be defined by the expression

$$\mathbf{a}(t) = a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j} + a_z(t)\,\mathbf{k}. \qquad (1)$$

The coordinates a_x, a_y, and a_z of the vector function $\mathbf{a}(t)$ are scalar functions of one variable t. To find out the geometrical meaning of the function $\mathbf{a}(t)$, we shall lay off the vectors $\mathbf{a}(t)$ for different values of t from the common origin O. With a change in the argument t in some interval, the terminal point of the vector $\mathbf{a}(t)$ will trace out a curve which is called the hodograph of the vector function $\mathbf{a}(t)$.

Definition. The locus of the extremities of the vectors $\mathbf{a}(t)$ issued from the common origin O as the argument t varies in some interval is called the hodograph.

The simplest example of a vector function is the radius vector $\mathbf{r}$ of a variable point with coordinates (x, y, z) dependent on time t:

$$\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

The hodograph of the vector function $\mathbf{r}(t)$ indicates the trajectory of the variable point.

Example. Find the vector function $\mathbf{r}(t)$ describing uniform anticlockwise rotation of a point round a circle of radius R.

△ We arrange the xOy Cartesian coordinate system so as to bring its origin into coincidence with the centre of the circle. The coordinates of the radius vector $\mathbf{r} = \{x, y\}$ of the rotating point are computed by the formulas $x = R \cos \varphi$, $y = R \sin \varphi$, where φ is the angle of rotation of the radius vector. In uniform anticlockwise rotation the angle of rotation is proportional to the rotation time: $\varphi = \omega t$, where t is the rotation time and ω the proportionality factor called the angular velocity of rotation. Thus, we have found that

$$\mathbf{r} = R \cos \omega t \cdot \mathbf{i} + R \sin \omega t \cdot \mathbf{j}. \blacktriangle$$

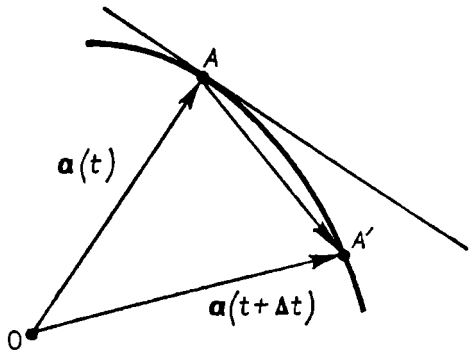

Fig. 155

2. The Derivative of a Vector Function of a Scalar Argument. Consider the vector function $\mathbf{a} = \mathbf{a}(t)$. Let us take two values of the argument t and $t + \Delta t$. We find the corresponding values of the vector function $\mathbf{a}(t)$ and $\mathbf{a}(t + \Delta t)$ and write the increment of the vector:

$$\Delta \mathbf{a} = \mathbf{a}(t + \Delta t) - \mathbf{a}(t).$$

This difference can be represented by a directed line segment $\overrightarrow{AA'}$ (Fig. 155), where the point A is the terminus of the vector $\mathbf{a}(t)$, and the point A' the terminus of the vector $\mathbf{a}(t + \Delta t)$.

We form the difference ratio

$$\frac{\Delta \mathbf{a}}{\Delta t} = \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t}$$

and pass in it to the limit as $\Delta t \to 0$.

If this limit exists, then it is called the derivative of the vector function $\mathbf{a} = \mathbf{a}(t)$ and is denoted by $\frac{d\mathbf{a}}{dt}$ or $\mathbf{a}'(t)$.

Consider, for instance, the radius vector $\mathbf{r}(t)$ of a moving point A. The increment in the radius vector

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

will be the displacement vector during time Δt, the difference ratio $\frac{\Delta \mathbf{r}}{\Delta t}$ will be the average-velocity vector during the time interval Δt. The instantaneous velocity of the point A is defined as the limit of the difference ratio as $\Delta t \to 0$:

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x \cdot \mathbf{i} + \Delta y \cdot \mathbf{j} + \Delta z \cdot \mathbf{k}}{\Delta t}.$$

Thus, the velocity $\mathbf{v}$ is the derivative of the vector function $\mathbf{r} = \mathbf{r}(t)$:

$$\mathbf{v} \frac{d\mathbf{r}}{dt} = x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}. \quad (1)$$

Example. Find the velocity vector of a point uniformly rotating anticlockwise in a circle of radius R.

△ As it was shown in the example from the preceding subsection, the radius vector $\mathbf{r}(t)$ of a rotating point is equal to

$$\mathbf{r} = R\cos\omega t\cdot\mathbf{i} + R\sin\omega t\cdot\mathbf{j}.$$

We compute the velocity $\mathbf{v}$ by formula (1):

$$\mathbf{v} = (R\cos\omega t)'\,\mathbf{i} + (R\sin\omega t)'\,\mathbf{j} = -\omega R\sin\omega t\cdot\mathbf{i} + \omega R\cos\omega t\cdot\mathbf{j}.$$

Hence, $|\mathbf{v}| = \sqrt{\omega^2R^2\sin^2\omega t + \omega^2R^2\cos^2\omega t} = \omega R$. The velocity vector $\mathbf{v}$ is perpendicular to the radius vector $\mathbf{r}$ since their scalar product is equal to zero:

$$\mathbf{rv} = -\omega R^2\sin\omega t\cos\omega t + \omega R^2\sin\omega t\cos\omega t = 0.$$

Thus, the velocity vector $\mathbf{v}$ is tangent to the circle. ▲

Let us find out the geometrical meaning of the derivative of the vector function $\mathbf{a} = \mathbf{a}(t)$. The difference ratio

$$\frac{\mathbf{a}(t+\Delta t)-\mathbf{a}(t)}{\Delta t} = \frac{\overrightarrow{AA'}}{\Delta t}$$

will be in the direction of the chord $\overrightarrow{AA'}$ (see Fig. 155). It is assumed that the increment in the argument Δt is positive. As $\Delta t \to 0$, the direction of the ratio $\frac{\Delta\mathbf{a}}{\Delta t} = \frac{\overrightarrow{AA'}}{\Delta t}$ will tend to the direction of the tangent line to the hodograph of the vector $\mathbf{a}(t)$. Thus, the derivative of the vector function $\mathbf{a}(t)$ coincides with the direction of the tangent line to the hodograph of the function $\mathbf{a}(t)$.

Let the terminus of the radius vector $\mathbf{r}(t)$ of the point M describe a certain curve. The position of the point M on the curve is defined by the length s of the arc counted from some fixed point M_0 to the point M. The radius vector is thus the function of the variable s: $\mathbf{r} = \mathbf{r}(s)$.

The derivative of the radius vector with respect to the variable s is directed, as it was cleared up above, along the tangent line to the trajectory described by the point M, that is, to the hodograph of the vector function $\mathbf{r} = \mathbf{r}(s)$.

The modulus of the derivative $\frac{d\mathbf{r}}{ds}$ is equal to the limit of the modulus of the difference ratio

$$\lim_{\Delta s\to 0}\left|\frac{\Delta\mathbf{r}}{\Delta s}\right|,$$

that is, to the limit of the ratio of the length of the chord $|\Delta \mathbf{r}|$ to the arc length Δs subtended by it. As is known, this limit is equal to unity (see 10.4, subsection 1).

Thus, the derivative of the radius vector $\frac{d\mathbf{r}}{ds}$ is the unit vector tangent to the curve $\mathbf{r} = \mathbf{r}(s)$ and directed towards the increase of the parameter s.

3. The Directional Derivative, Gradient, and Potential of a Vector Field. For the purpose of studying scalar fields, we introduce the notions of directional derivative and gradient.

Let a scalar field $f(\mathbf{r}) = f(x, y, z)$ be given. We fix a point $M(\mathbf{r})$ in this field and draw through it a straight line in the direction of the given unit vector $\mathbf{s}$. We take on this straight line a neighbouring point $M_1(\mathbf{r} + \varepsilon \cdot \mathbf{s})$ defined by the scalar variable ε. Let Δf denote the increment in the function $f(\mathbf{r})$. We form the ratio of the increment in the function Δf to ε and then find the limit of this ratio as $\varepsilon \to 0$.

The limit of the ratio $\frac{\Delta f}{\varepsilon}$ as $\varepsilon \to 0$ is called the (directional) derivative of the scalar field $f(x, y, z)$ in the direction of $\mathbf{s}$ at the point M and is denoted by $\frac{\partial f}{\partial \mathbf{s}}$, that is,

$$\frac{\partial f}{\partial \mathbf{s}} = \lim_{\varepsilon \to 0} \frac{\Delta f}{\varepsilon}. \tag{1}$$

To compute the derivative in the direction of $\mathbf{s}$, let us consider the direction cosines of the unit vector $\mathbf{s}$ which are equal to its projections onto the axes of a rectangular coordinate system

$$s_x = \cos(\widehat{\mathbf{s}, \mathbf{i}}), \quad s_y = \cos(\widehat{\mathbf{s}, \mathbf{j}}), \quad s_z = \cos(\widehat{\mathbf{s}, \mathbf{k}}).$$

The increment of the function $f(\mathbf{r})$ is expressed in terms of the direction cosines and quantity ε as follows:

$$\begin{aligned} \Delta f &= f(\mathbf{r} + \varepsilon \mathbf{s}) - f(\mathbf{r}) \\ &= f(x + \varepsilon \cos(\widehat{\mathbf{s}, \mathbf{i}}),\ y + \varepsilon \cos(\mathbf{s}, \mathbf{j}),\ z \\ &\quad + \varepsilon \cos(\widehat{\mathbf{s}, \mathbf{k}})) - f(x, y, z). \end{aligned} \tag{2}$$

Substituting expression (2) for the increment Δf into equality (1), we get:

$$\begin{aligned} \frac{\partial f}{\partial \mathbf{s}} &= \lim_{\varepsilon \to 0} \frac{\Delta f}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon \cos(\widehat{\mathbf{s}, \mathbf{i}}),\ y + \varepsilon \cos(\widehat{\mathbf{s}, \mathbf{j}}),\ z + \varepsilon \cos(\widehat{\mathbf{s}, \mathbf{k}})) - f(x, y, z)}{\varepsilon}. \end{aligned} \tag{3}$$

The right-hand side of equality (3) is the derivative of a composite function with respect to the independent variable ε. Using the formula for computing composite functions (see 10.2, subsection 5), we arrive at the desired formula for computing the derivative of the function $f(\mathbf{r})$ in the direction of s:

$$\frac{\partial f}{\partial \mathbf{s}} = \frac{\partial f}{\partial x} \cdot \cos(\widehat{\mathbf{s}, \mathbf{i}}) + \frac{\partial f}{\partial y} \cdot \cos(\widehat{\mathbf{s}, \mathbf{j}}) + \frac{\partial f}{\partial z} \cdot \cos(\widehat{\mathbf{s}, \mathbf{k}}). \quad (4)$$

Let us now introduce the notion of the gradient of a scalar field $f(\mathbf{r})$. The gradient of the scalar field $f(\mathbf{r}) = f(x, y, z)$ at a point $M(\mathbf{r})$ is the vector whose projections onto the axes of a rectangular coordinate system are equal to $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

The gradient of the scalar $f(x, y, z)$ is denoted by the symbol grad f. Thus,

$$\operatorname{grad} f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z}. \quad (5)$$

The modulus of the vector grad f is equal to

$$|\operatorname{grad} f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}.$$

The projection of grad f onto the direction of the vector s is equal to the scalar product of these vectors. In field theory the scalar product is most frequently denoted in the following way: (grad f, s). Henceforward, we shall use this notation.

Expanding the scalar product (grad f, s) in terms of the coordinates of the vectors grad f and s, we find that

$$(\operatorname{grad} f, \mathbf{s}) = \frac{\partial f}{\partial x} \cos(\widehat{\mathbf{s}, \mathbf{i}}) + \frac{\partial f}{\partial y} \cos(\widehat{\mathbf{s}, \mathbf{j}}) + \frac{\partial f}{\partial z} \cos(\widehat{\mathbf{s}, \mathbf{k}}).$$

Comparing this formula with formula (4), we find:

$$(\operatorname{grad} f, \mathbf{s}) = \frac{\partial f}{\partial \mathbf{s}}, \quad (6)$$

that is, the projection of the gradient on an arbitrary direction of s is equal to the derivative of the function $f(\mathbf{r})$ in the direction of s at a given point.

Since

$$(\operatorname{grad} f, \mathbf{s}) = |\operatorname{grad} f|\,|\mathbf{s}| \cos(\widehat{\operatorname{grad} f, \mathbf{s}}) = |\operatorname{grad} f| \cos(\widehat{\operatorname{grad} f, \mathbf{s}}),$$

the derivative $\frac{\partial f}{\partial \mathbf{s}}$ attains the greatest value for the direction of s just coinciding with the direction of grad f. In this case, the

magnitude of the derivative $\frac{\partial f}{\partial s}$ will be equal to

$$\frac{\partial f}{\partial s} = |\operatorname{grad} f|. \tag{7}$$

As is seen from equality (7), the derivative $\frac{\partial f}{\partial s}$ is positive in this case.

Thus, the gradient of the function $f(x, y, z)$ indicates the direction of the fastest increase in the function f, being numerically equal to the derivative in this direction.

Example 1. Find the gradient of the function

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}.$$

△ To compute grad r, we find the partial derivatives of the function $r = r(x, y, z)$ with respect to x, y, and z:

$$\frac{\partial r}{\partial x} = \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}},$$

$$\frac{\partial r}{\partial y} = \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}},$$

$$\frac{\partial r}{\partial z} = \frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Thus,

$$\operatorname{grad} r = \frac{x-x_0}{r}\mathbf{i} + \frac{y-y_0}{r}\mathbf{j} + \frac{z-z_0}{r}\mathbf{k}. \ ▲$$

Example 2. Find the gradient of the function $f(x, y, z) = (\mathbf{a}, \mathbf{r})$, where $\mathbf{a} = \{a_x, a_y, a_z\}$ is a constant vector, and $\mathbf{r} = \{x, y, z\}$.

△ We compute the partial derivatives of the function $f(x, y, z) = a_x \cdot x + a_y \cdot y + a_z \cdot z$:

$$\frac{\partial f}{\partial x} = a_x, \quad \frac{\partial f}{\partial y} = a_y, \quad \frac{\partial f}{\partial z} = a_z.$$

Thus,

$$\operatorname{grad}(\mathbf{a}, \mathbf{r}) = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \mathbf{a}. \ ▲$$

The notion of potential vector field is tightly connected with that of gradient. The vector $\mathbf{a}(x, y, z)$, which is the gradient of the scalar $f(x, y, z)$, is called the potential vector, and the function $f(x, y, z)$ is called the potential of the given vector for which

$$\mathbf{a} = \operatorname{grad} f.$$

In order to simplify the computations of the gradient of complicated expressions, we introduce the "nabla" operator (or Hamil-

tonian):

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

The operator ∇ is applied to scalar functions according to the following rule:

$$\nabla f(x, y, z) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}.$$

Let us give without proof the basic formulas for computing gradients with the help of the "nabla" operator:

(a) $\nabla(f + \varphi) = \nabla f + \nabla\varphi$;

(b) $\nabla(Cf) = C\nabla f$, where C is a constant quantity;

(c) $\nabla(\varphi f) = \varphi\nabla f + f\nabla\varphi$;

(d) $\nabla f(\varphi) = f'_{\varphi} \cdot \nabla\varphi$.

Exercises

1. Compute the derivative of the function $f = x^2 + y^2 + z^2$ in the direction of the vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

2. Compute the derivative of the function $f = x + y^2 - 3xz$ in the direction of the vector $\mathbf{k}$.

3. Compute the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^2 + z^2$;

(b) $f(x, y, z) = x + 3y + z$;

(c) $f(x, y, z) = x^2 - y^2 - z^2$.

4. Compute the gradients of the functions r^2, $\frac{1}{r}$, r^h, where $r = \sqrt{x^2 + y^2 + z^2}$.

5. Find the direction of the fastest increase in the function $f(x, y, z) = x^2 - y^2 + x + 3y + z$.

12.5. Line Integrals

1. The Line Integral of the First Kind. Let three functions of the independent variable t: $x = \varphi(t)$, $y = f(t)$, $z = \psi(t)$ be given, which are differentiable in the interval (t_1, t_2) and continuous on the closed interval $[t_1, t_2]$. When t increases from t_1 to t_2, the point $M(x, y, z)$ describes an arc AB of a curve AB (closed or open). Under our assumptions, the curve AB will have a finite length L. Let s denote the arc length of the curve enclosed between a fixed point (one of the endpoints of the curve AB) and the variable point M. We take s as a parameter which characterizes the position of the

variable point M on the curve AB. The coordinates of the point M will be functions of the parameter s:

$$x = x(s), \quad y = y(s), \quad z = z(s),$$

and s varies in the limits from 0 to L, where L is the length of the curve AB.

Let the function $P(M)$ be defined on the curve AB as a function of the point M. The function $P(M)$ is a function of the coordinates of the point M: $P(M) = P(x, y, z)$.

Since the coordinates of the point M are, in turn, functions of the parameter s, the function $P(M)$ is a composite function of the variable s: $P(M) = P(x(s), y(s), z(s))$. Further, we assume that the function $P(x(s), y(s), z(s))$ is continuous for all $s \in [0, L]$, and give the definition of the line integral of the first kind.

Definition. The line integral of the first kind of the function $P(x, y, z)$ over the arc AB is the definite integral of the function $P(x(s), y(s), z(s))$ with respect to the variable s between the limits 0 and L.

The line integral of the first kind is denoted by the symbol

$$\int_{AB} P(x, y, z)\, ds \text{ or } \int_{C} P(x, y, z)\, ds,$$

where A and B denote, respectively, the initial and final points of the curve over which the integral is taken, and C denotes the entire curve. Thus, by definition, the following equality is valid:

$$\int_{AB} P(x, y, z)\, ds = \int_{0}^{L} P(x(s), y(s), z(s))\, ds. \tag{1}$$

The right-hand side of equality (1) is an ordinary definite integral with respect to the variable s between the limits 0 and L. Since the arc length is regarded as positive irrespective of the endpoint it is counted from, the line integral of the first kind is independent of the direction in which the curve C is traced.

To find out the physical meaning of the line integral of the first kind, let us consider the problem on computing the mass of a material thread C. Let the thread C have a mass with line density $P(x, y, z)$, $(x, y, z) \in C$. Then the element of the curve C corresponding to the interval $[s, s + ds]$ of variation of s has a mass equal to

$$P(x, y, z)\, ds.$$

Hence it follows that the mass M of the entire curve C will be equal to

$$M = \int_{0}^{L} P(x, y, z)\, ds = \int_{C} P(x, y, z)\, ds. \tag{2}$$

Thus, the mass of the material thread C is equal to the line integral of the first kind over the curve C of the line density of mass distribution $P(x, y, z)$, $(x, y, z) \in C$.

If the curve C is represented parametrically, then its coordinates are specified as functions of the parameter t:

$$x = \varphi(t), \quad y = f(t), \quad z = \psi(t), \quad t_1 \leqslant t \leqslant t_2.$$

For the modulus of the increment in the radius vector $\mathbf{r} = \{x, y, z\}$ of the points of the curve C the following equality is valid:

$$|\Delta \mathbf{r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

Then

$$\frac{\Delta s}{\Delta t} = \frac{\Delta s}{|\Delta \mathbf{r}|} \cdot \frac{|\Delta \mathbf{r}|}{\Delta t} = \frac{\Delta s}{|\Delta \mathbf{r}|} \cdot \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}.$$

Passing to the limit in this equality as $\Delta t \to 0$ and taking into account that $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{|\Delta \mathbf{r}|} = \lim\limits_{\Delta s \to 0} \frac{\Delta s}{|\Delta \mathbf{r}|} = 1$ (see subsection 2 of the preceding section), we get

$$\frac{ds}{dt} = \sqrt{(x')^2 + (y')^2 + (z')^2} = \sqrt{(\varphi')^2 + (f')^2 + (\psi')^2}.$$

Then the differential of the variable length of the arc s is equal to

$$ds = s'(t)\, dt = \sqrt{(\varphi')^2 + (f')^2 + (\psi')^2}\, dt.$$

Changing in definite integral (1) to the variable of integration t, we get the following formula for computing the line integral:

$$\int_C P(x, y, z)\, ds = \int_{t_1}^{t_2} P(\varphi(t), f(t), \psi(t)) \sqrt{(\varphi')^2 + (f')^2 + (\psi')^2}\, dt. \quad (3)$$

For formula (3) to be valid, it is necessary that the derivatives φ', f', ψ' be continuous. Also note that in the integral on the right-hand side of equality (3) t_2 must always be greater than t_1.

Example 1. Compute the line integral $\int\limits_C (x^2 + y^2)\, ds$ round the circle given parametrically by the formulas

$$x = a \cos t, \quad y = a \sin t, \quad 0 \leqslant t \leqslant 2\pi.$$

△ We compute the given line integral by formula (3):

$$\int_C P(x, y)\, ds = \int_0^{2\pi} (a^2 \cos^2 t + a^2 \sin^2 t) \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt = 2\pi a^3. \blacktriangle$$

Example 2. Find the mass of the helix

$$x = a\cos t, \quad y = a\sin t, \quad z = ht, \quad 0 \leqslant t \leqslant 2\pi,$$

whose density is $P(x, y, z) = kz$.

△ The mass M is determined by formula (2). Computations are performed by formula (3):

$$M = \int_C P(x, y, z)\,ds = \int_0^{2\pi} kht\sqrt{a^2\sin^2 t + a^2\cos^2 t + h^2}\,dt$$

$$= kh\sqrt{a^2+h^2}\cdot\frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2 kh\sqrt{a^2+h^2}. \ \blacktriangle$$

If the curve C is given explicitly in the plane by the equation $y = f(x)$, $x \in [x_1, x_2]$, then, taking the variable x as a parameter, we get:

$$\int_C P(x, y)\,ds = \int_{x_1}^{x_2} P(x, f(x))\sqrt{(1+f'(x))^2}\,dx. \qquad (4)$$

Example 3. Evaluate the line integral $\int_C xy\,ds$, where C is the part of the parabola $y = x^2$, $0 \leqslant x \leqslant \sqrt{3}/2$.

△ Since the curve C is represented explicitly, we make use of formula (4):

$$\int_C xy\,ds = \int_0^{\sqrt{3}/2} x^3\sqrt{1+(2x)^2}\,dx = \int_1^4 \frac{t-1}{4}\cdot\sqrt{t}\cdot\frac{dt}{8}$$

$$= \frac{1}{32}\left(\frac{2}{5}t^{5/2}\Big|_1^4 - \frac{2}{3}t^{3/2}\Big|_1^4\right) = \frac{29}{120},$$

where we have used the substitution $t = 1 + 4x^2$ for the evaluation of the given integral. ▲

2. The Line Integral of the Second Kind. Consider a continuous smooth curve C in space represented parametrically by the functions

$$x = \varphi(t), \quad y = f(t), \quad z = \psi(t), \quad t_1 \leqslant t \leqslant t_2.$$

Suppose that the positive direction is indicated on the curve C. According to the direction chosen, we denote the initial point by A_0 and the final point by A_1.

Let on the curve C a vector field be given:

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k},$$

where P, Q, R are continuous functions defined on the curve C. At each point of the curve C we draw a tangent unit vector τ (Fig. 156), the direction τ corresponding to the direction of motion from A_0 to A_1.

The scalar product $(\mathbf{a}, \tau)$ depends on the position of the point M on the curve C, that is, it is the function of the variable s, where s is the length of the arc A_0M.

Let us now give the definition of the line integral of the second kind.

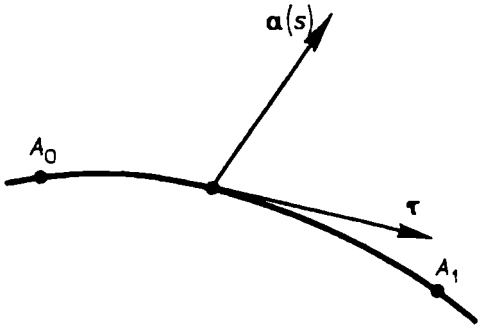

Fig. 156

Definition. The line integral of the second kind of a vector function $\mathbf{a}(x, y, z)$ along a curve C in the direction from A_0 to A_1 is the following definite integral with respect to the variable s taken between 0 and L:

$$\int_0^L (\mathbf{a}, \tau)\, ds, \tag{1}$$

where L is the length of the arc over which the integration is performed.

The line integral of the second kind is denoted by

$$\int_C (\mathbf{a}, d\mathbf{r}) \text{ or } \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz.$$

The first of these notations is called the vector notation, and the second the coordinate notation.

Thus, by definition, the following equalities are valid:

$$\int_C (\mathbf{a}, d\mathbf{r}) = \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dr = \int_0^L (\mathbf{a}, \tau)\, ds.$$

From the definition of the line integral of the second kind it follows that, as the direction of tracing the curve C is reversed, the sign of this integral changes. Indeed, if we choose A_1 as the initial point, and A_0 as the terminal point (see Fig. 156), then the only tangent vector conforming with the direction of tracing the curve in this case will be the vector $-\tau$, and the sign of integral (1) will reverse.

Many physical problems lead to line integrals of the second kind, for example, the problem on the work done by the force $\mathbf{F} = \mathbf{F}(x, y, z)$ along the curve C.

To compute the line integral of the second kind, we pass over to the variable of integration t:

$$\int_0^L (\mathbf{a}, \boldsymbol{\tau})\, ds = \int_0^L \left(\mathbf{a}, \frac{d\mathbf{r}}{ds}\right) ds = \int_0^L \left(\mathbf{a}, \frac{d\mathbf{r}}{dt}\frac{dt}{ds}\right) ds = \int_{t_1}^{t_2} \left(\mathbf{a}, \frac{d\mathbf{r}}{dt}\right) dt. \quad (2)$$

The derivative of the vector function $\mathbf{r} = \mathbf{r}(t)$ with respect to the argument t is equal to (see 12.4, subsection 2):

$$\frac{d\mathbf{r}}{dt} = \varphi'\mathbf{i} + f'\mathbf{j} + \psi'\mathbf{k}.$$

Expanding the scalar product $\left(\mathbf{a}, \frac{d\mathbf{r}}{dt}\right)$ in expression (2), we get the formula for computing the line integral:

$$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = \int_0^L (\mathbf{a}, \boldsymbol{\tau})\, ds$$

$$= \int_{t_1}^{t_2} [P(\varphi(t), f(t), \psi(t))\, \varphi'(t) + Q(\varphi(t), f(t), \psi(t))\, f'(t)$$

$$+ R(\varphi(t), f(t), \psi(t))\, \psi'(t)]\, dt. \quad (3)$$

The value of the parameter $t = t_1$ must correspond to the initial point of the curve C, and the value $t = t_2$ to the terminal point. Here both cases are possible: $t_2 > t_1$ and $t_2 < t_1$.

Formula (3) is also obtained directly from the coordinate notation of the line integral of the second kind: we only have to substitute $\varphi(t)$, $f(t)$, and $\psi(t)$ for x, y, and z, respectively, and to take into account that $d\varphi(t) = \varphi'(t)\, dt$, etc. This is why this notation of the line integral of the second kind is considered to be successful and is most widely used in mathematical literature. In physical literature, a brief vector notation of this integral is used most frequently.

If the curve C is given in the plane by the equation $y = f(x)$, $x \in [x_1, x_2]$, then, taking the variable x as a parameter, we get

$$\int_C P(x, y)\, dx + Q(x, y)\, dy = \int_{x_1}^{x_2} [P(x, f(x)) + Q(x, f(x))\, f'(x)]\, dx. \quad (4)$$

Example 1. Evaluate the line integral of the second kind $\int_C x\, dx + xy\, dy$ along the rectilinear segment from the point (0, 0) to the point (1, 2).

△ The equation of the straight line passing through the points (0, 0) and (1, 2) is $y = 2x$. For the line segment whose endpoints are (0, 0) and (1, 2) the variable x changes from zero to unity.

To evaluate the line integral of the second kind, we make use of formula (4), setting $y = f(x) = 2x$:

$$\int_C x\,dx + xy\,dy = \int_0^1 (x + 4x^2)\,dx = \left(\frac{x^2}{2} + \frac{4}{3}x^3\right)\Big|_0^1 = \frac{11}{6}. \blacktriangle$$

If C is a closed curve, then the line integral is written in the form

$$\oint_C (\mathbf{a}, d\mathbf{r}) \quad \text{or} \quad \oint_C P\,dx + Q\,dy + R\,dz.$$

If there are no explicit directions as to the direction of tracing the closed curve C, then it is assumed that the curve C is traced so that the domain lying inside the contour is kept on the left of the point moving round the contour.

Example 2. Evaluate the integral of the second kind

$$\oint_C y\,dx - x\,dy$$

round a circle of radius R_0 centred at the origin and oriented clockwise.

△ The circle in parametric form is given by the functions

$$x = R_0 \cos t, \quad y = R_0 \sin t, \quad 0 \leqslant t \leqslant 2\pi.$$

Using formula (3) for computing the line integral of the second kind, we set in it the function $R = 0$. Let us also take into consideration that since the curve C is traced clockwise, the initial point should be assigned the value of the parameter $t = 2\pi$, and the terminal point the value $t = 0$.

$$\oint_C y\,dx - x\,dy = -\int_{2\pi}^{0} (R_0^2 \sin^2 t + R_0^2 \cos^2 t)\,dt$$

$$= R_0^2 \int_0^{2\pi} dt = 2\pi R_0^2. \blacktriangle$$

3. The Riemann-Green Formula. Consider in the xy-plane the domain D together with its boundary C (Fig. 157).

Let then in a closed domain D continuous functions $P(x, y)$ and $Q(x, y)$ be defined together with their derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$.

Consider a double integral over the domain D

$$\iint\limits_D \frac{\partial P}{\partial y}\, dx\, dy. \tag{1}$$

We transform the double integral (1) to an iterated one (see Fig. 157):

$$\iint\limits_D \frac{\partial P}{\partial y}\, dx\, dy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\, dy$$

$$= \int\limits_a^b P(x,\ y_2(x))\, dx - \int\limits_a^b P(x,\ y_1(x))\, dx. \tag{2}$$

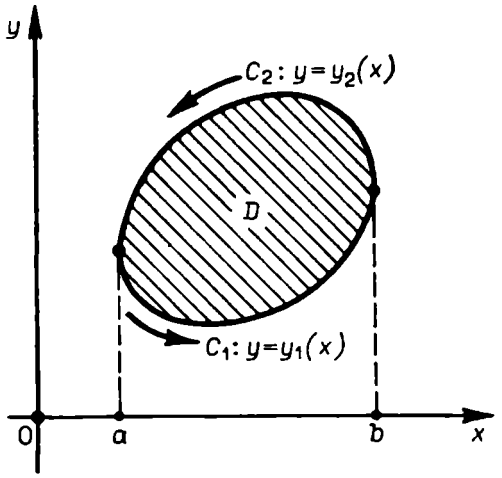

Fig. 157

The integrals in formula (2) are, obviously, equal to the line integrals of the second kind along the directed arcs C_1 and C_2, respectively:

$$\begin{aligned} \int\limits_a^b P(x,\ y_1(x))\, dx &= \int\limits_{C_1} P(x,\ y)\, dx, \\ \int\limits_a^b P(x,\ y_2(x))\, dx &= -\int\limits_{C_2} P(x,\ y)\, dx. \end{aligned} \tag{3}$$

We then substitute integrals (3) into formula (2):

$$\iint\limits_D \frac{\partial P}{\partial y}\, dx\, dy = -\int\limits_{C_1} P(x,\ y)\, dx - \int\limits_{C_2} P(x,\ y)\, dx. \tag{4}$$

The integrals on the right-hand side of equality (4) can be combined into one integral round a closed contour C since C is a com-

bination of the directed arcs C_1 and C_2:

$$\iint\limits_D \frac{\partial P}{\partial y}\,dx\,dy = -\oint\limits_C P(x,\ y)\,dx. \tag{5}$$

We now consider the integral of $\frac{\partial Q(x,\ y)}{\partial x}$ over the domain D:

$$\iint\limits_D \frac{\partial Q}{\partial x}\,dx\,dy,$$

and transform it to the form

$$\iint\limits_D \frac{\partial Q}{\partial x}\,dx\,dy = \oint\limits_C Q\,dy. \tag{6}$$

Subtracting both sides of equality (5) from those of equality (6) we get the formula

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint\limits_C P\,dx + Q\,dy,$$

which is called the Riemann-Green formula. It is widely used in the field theory.

Exercises

1. Evaluate the following line integrals of the first kind:

(a) $\int\limits_C xy\,ds$, where C is a rectangle with vertices $A_1(0,\ 0)$, $A_2(4,\ 0)$, $A_3(4,\ 2)$, and $A_4(0,\ 2)$;

(b) $\int\limits_C y\,ds$, where C is a part of the parabola $y = 2\sqrt{x}$, $0 \leqslant x \leqslant 1$;

(c) $\int\limits_C (x - y)\,ds$, where C is a circle $x^2 + y^2 = 2x$.

2. Evaluate the following line integrals of the second kind:

(a) $\oint\limits_C x\,dy$, where C is a triangle formed by the coordinate axes and the straight line $x + y = 2$ which is traced anticlockwise;

(b) $\oint\limits_C (x + y)\,dx + y^2x\,dy$, where C is a circle of radius $R_0 = 2$ with centre at the origin which is traced anticlockwise.

12.6. Surface Integrals of the First Kind

1. A Surface in Space. Let a surface in space be given. Let us denote it by Ω. We shall specify the position of the surface in a given rectangular coordinate system $Oxyz$ (Fig. 158).

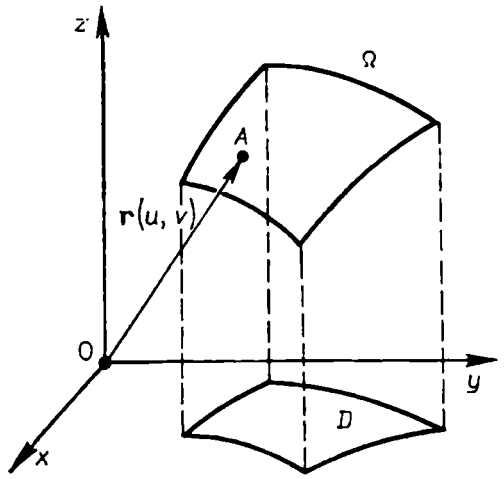

Fig. 158

Suppose that straight lines parallel to the z-axis pierce the surface only at one point. In this case, the equation of the surface can be written either in explicit form

$$z = f(x, y) \qquad (1)$$

or in vector form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\,\mathbf{k}, \qquad (2)$$

where $\mathbf{r}$ is the radius vector drawn to the point A on the surface $A(x, y, f(x, y))$ (see Fig. 158).

We see that the radius vector $\mathbf{r}$ of points on the surface is a function of two independent variables x and y which are the coordinates of the points of the domain D lying in the xy-plane.

In the general case, the surface is defined by the vector equation

$$\mathbf{r} = \varphi(u, v)\,\mathbf{i} + f(u, v)\,\mathbf{j} + \psi(u, v)\,\mathbf{k}. \qquad (3)$$

The radius vector $\mathbf{r}$ in equation (3) is a function of the variables u and v. The set of number pairs (u, v) belongs to some domain D_1.

The functions $\varphi(u, v)$, $f(u, v)$, and $\psi(u, v)$ are supposed to be continuous together with their partial derivatives. In addition, it is assumed that each number pair corresponds uniquely to a point of the surface Ω defined by the radius vector $\mathbf{r}$ by formula (3), and vice versa, each point of the surface Ω corresponds to one pair (u, v) belonging to the domain D_1.

For instance, a circular cylindrical surface of radius R_0 and height H is defined by the vector equation

$$\mathbf{r} = R_0 \cos u \cdot \mathbf{i} + R_0 \sin u \cdot \mathbf{j} + v \cdot \mathbf{k},$$

where u denotes the polar angle φ in the xy-plane, and v the z-coordinate. The set of number pairs (u, v) belongs to the rectangle D_1: $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant H$. The functions $\varphi(u) = R_0 \cos u$, $f(u) = R_0 \sin u$, and $\psi(v) = v$ are continuous together with their derivatives in the domain D_1.

2. Surface Area. If the surface Ω is represented parametrically by equation (3) from the preceding subsection, then its area is de-

fined as a number equal to the double integral of the modulus of the vector product $\mathbf{r}'_u$ and $\mathbf{r}'_v$:

$$S = \iint\limits_{D_1} |[\mathbf{r}'_u, \mathbf{r}'_v]| \, du \, dv. \tag{1}$$

To explain the geometrical meaning of the definition of the area S of the surface Ω, let us represent the double integral (1) as the limit of integral sums

$$\iint\limits_{D_1} |[\mathbf{r}'_u, \mathbf{r}'_v]| \, du \, dv = \lim_{n \to \infty} \sum_{i=1}^{n} |[\mathbf{r}'_u(u_i, v_i), \mathbf{r}'_v(u_i, v_i)]| \, \Delta u_i \, \Delta v_i. \tag{2}$$

Consider an individual term in integral sum (2):

$$\Delta\omega_i = |[\mathbf{r}'_u, \mathbf{r}'_v]| \, \Delta u_i \, \Delta v_i.$$

Let us bring the factors Δu_i and Δv_i under the sign of the vector product:

$$\Delta\omega_i = |[\mathbf{r}'_u \, \Delta u_i, \mathbf{r}'_v \, \Delta v_i]|. \tag{3}$$

The products $\mathbf{r}'_u \Delta u_i$ and $\mathbf{r}'_v \Delta v_i$ are partial differentials of the vector function (u, v) at the point (u_i, v_i), that is,

$$\mathbf{r}'_u \, \Delta u_i = d_u\mathbf{r}(u_i, v_i) \quad \text{and} \quad \mathbf{r}'_v \, \Delta v_i = d_v\mathbf{r}(u_i, v_i). \tag{4}$$

Substituting expressions (4) into equality (3), we get

$$\Delta\omega_i = |[d_u\mathbf{r}, d_v\mathbf{r}]|. \tag{5}$$

To geometrically interpret formula (5), we consider on the surface Ω the lines passing through the point A with radius vector $\mathbf{r}(u_i, v_i)$ and defined by the vector equations

$$\mathbf{r} = \mathbf{r}(u_i, v), \quad \mathbf{r} = \mathbf{r}(u, v_i). \tag{6}$$

The derivatives $\mathbf{r}'_u(u_i, v_i)$ and $\mathbf{r}'_v(u_i, v_i)$ are tangent to curves (6) at the point A (see 10.4, subsection 1). The partial differentials

$$d_u\mathbf{r} = \mathbf{r}'_u \, \Delta u_i \quad \text{and} \quad d_v\mathbf{r} = \mathbf{r}'_v \, \Delta v_i$$

are directed line segments $\overrightarrow{AB}$ and $\overrightarrow{AD}$ (Fig. 159) of the tangents to the lines (6) at the point A. We construct a parallelogram $ABCD$. It is found in the tangent plane to the surface Ω at the point A. The area of the parallelogram $ABCD$ will be equal to the modulus of the vector product of the vectors $d_u\mathbf{r}$ and $d_v\mathbf{r}$, that is, to

$$|d_u\mathbf{r} \times d_v\mathbf{r}|.$$

Hence it is seen that $\Delta\omega_i$ is numerically equal to the area of the parallelogram lying in the tangent plane to the surface Ω at the point A with sides $d_u\mathbf{r}$ and $d_v\mathbf{r}$.

The sum of all the areas $\Delta\omega_i$ is approximately equal to the area of the surface Ω. Therefore the limit in formula (2) has a geometrical meaning of the area S of the given surface Ω.

Let us consider the case of explicit representation of the surface Ω in a rectangular coordinate system (see formulas (1) and (2) in the

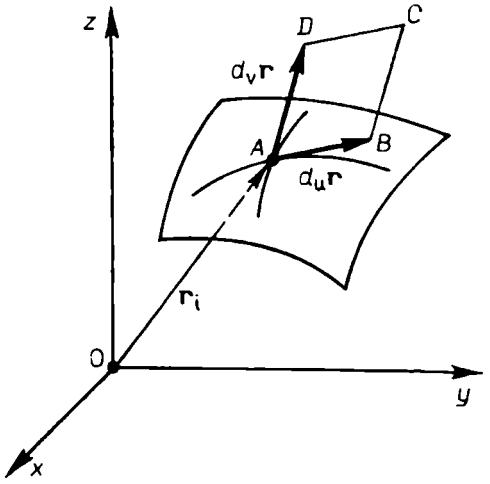

Fig. 159

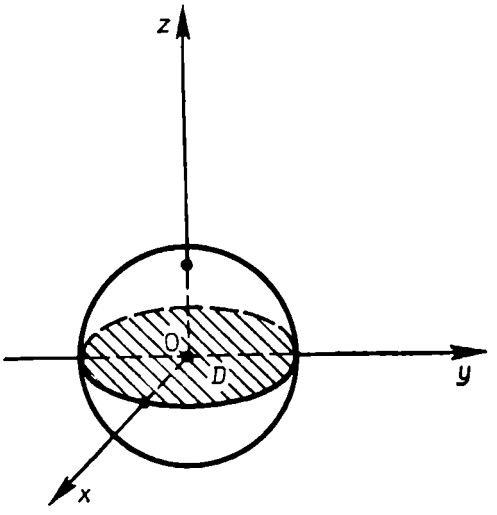

Fig. 160

preceding subsection) when the coordinates x and y serve as parameters u and v.

We compute the derivatives $\mathbf{r}'_x$ and $\mathbf{r}'_y$ of the function $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + f(x, y)\,\mathbf{k}$:

$$\mathbf{r}'_x = \mathbf{i} + \mathbf{k}f'_x, \quad \mathbf{r}'_y = \mathbf{j} + \mathbf{k}f'_y.$$

The vector product of the vectors $\mathbf{r}'_x$ and $\mathbf{r}'_y$ is found by expanding the third-order determinant in elements of the first row (see 1.5, subsection 3):

$$\mathbf{r}'_x \times \mathbf{r}'_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{vmatrix} = -\mathbf{i}f'_x - \mathbf{j}f'_y + \mathbf{k}.$$

The modulus of the vector product is equal to $\sqrt{1 + (f'_x)^2 + (f'_y)^2}$. Then it follows from formula (1) that the area S of the surface Ω is

$$S = \iint\limits_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, dx\, dy. \tag{7}$$

Example 1. Find the area of the surface of a sphere of radius R.

△ Suppose that the centre of the sphere coincides with the origin of the rectangular coordinate system $Oxyz$ (Fig. 160).

The equation of the surface of a sphere has the form $x^2 + y^2 + z^2 = R^2$. Consider a hemisphere defined by the condition $z \geqslant 0$.

For this hemisphere $z = \sqrt{R^2 - x^2 - y^2}$. We compute the area of the hemisphere by formula (7):

$$S = \iint\limits_D \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\, dx\, dy$$

$$= R \iint\limits_D \sqrt{\frac{1}{R^2 - x^2 - y^2}}\, dx\, dy.$$

Here the domain D is represented by a circle of radius R in the xy-plane with centre at the origin (see Fig. 160). We compute the double integral in polar coordinates:

$$S = R \iint\limits_D \frac{dx\, dy}{\sqrt{R^2 - x^2 - y^2}}$$

$$= R \iint\limits_D \frac{r\, dr\, d\varphi}{\sqrt{R^2 - r^2}} = -R \int\limits_0^{2\pi} d\varphi \int\limits_0^R d\sqrt{R^2 - r^2}$$

$$= -2\pi R \cdot \sqrt{R^2 - r^2}\,\Big|_{r=0}^{r=R} = 2\pi R^2.$$

Thus, the area of the hemisphere is equal to $2\pi R^2$. The area of the entire sphere S_s will be twice the area of the hemisphere, that is, $S_s = 4\pi R^2$. ▲

Example 2. Compute the area of the cylinder of radius R_0 and altitude H.

△ Suppose that the axis of the cylinder coincides with the z-axis (Fig. 161).

The cylinder is defined in cylindrical coordinates by the equations

$$x = R_0 \cos \varphi, \quad y = R_0 \sin \varphi,$$
$$0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant z \leqslant H.$$

Thus, the vector equation of the cylinder has the form:

$$\mathbf{r} = R_0 \cos \varphi \mathbf{i} + R_0 \sin \varphi \mathbf{j} + z\mathbf{k},$$

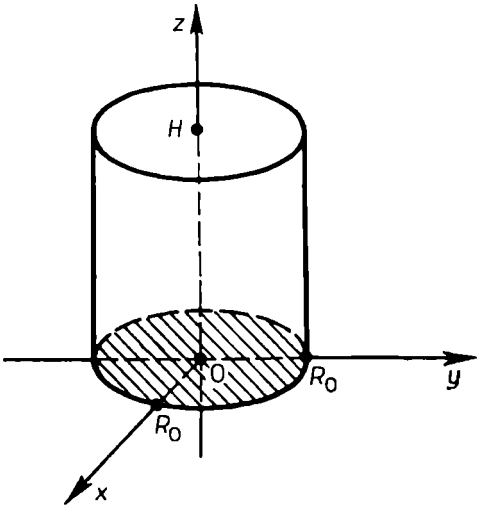

Fig. 161

where $\mathbf{r}$ is the radius vector of the points of the cylinder. In this case, the role of the parameters u and v (see formula (3) in subsection 1) is played by the variables φ and z. The set of number pairs (φ, z) belongs to the rectangle D_1: $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant H$.

We then compute the vector product of the vectors $\mathbf{r}'_\varphi = -R_0 \sin\varphi\,\mathbf{i} + R_0\cos\varphi\,\mathbf{j} + 0\cdot\mathbf{k}$ and $\mathbf{r}'_z = 0\cdot\mathbf{i} + 0\cdot\mathbf{j} + \mathbf{k}$:

$$\mathbf{r}'_\varphi \times \mathbf{r}'_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R_0\sin\varphi & R_0\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = R_0\cos\varphi\,\mathbf{i} + R_0\sin\varphi\,\mathbf{j} + 0\cdot\mathbf{k}.$$

Its modulus is equal to R_0. Substituting the modulus of the vector product in formula (1), we get that the area of the cylinder

$$S = \iint\limits_{D_1} R_0\,d\varphi\,dz = R_0\int\limits_0^{2\pi} d\varphi \int\limits_0^H dz = 2\pi R_0 H. \ \blacktriangle$$

3. The Definition of the Surface Integral of the First Kind. Suppose that the surface Ω is given by the vector equation

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \varphi(u, v)\,\mathbf{i} + f(u, v)\,\mathbf{j} + \psi(u, v)\,\mathbf{k},$$

where $(u, v) \in D_1$.

Let a function $F(x, y, z)$ be defined on the surface Ω. The surface integral of the first kind of the function $F(x, y, z)$ over the surface Ω is defined as the double integral

$$\iint\limits_{D_1} F(\varphi, f, \psi)\,|\mathbf{r}'_u \times \mathbf{r}'_v|\,du\,dv.$$

The surface integral of the first kind is denoted as follows:

$$\iint\limits_\Omega F(x, y, z)\,d\omega.$$

Thus, by definition, the following equality is valid:

$$\iint\limits_\Omega F(x, y, z)\,d\omega = \iint\limits_{D_1} F(\varphi(u, v), f(u, v), \psi(u, v))\,|\mathbf{r}'_u \times \mathbf{r}'_v|\,du\,dv. \tag{1}$$

In the case of explicit representation of the surface Ω by the equation $z = f(x, y)$, $(x, y) \in D$, in rectangular Cartesian coordinates we make use of the result for $\mathbf{r}'_x \times \mathbf{r}'_y$ obtained in the preceding subsection. Then equality (1) takes the form:

$$\iint\limits_\Omega F(x, y, z)\,d\omega = \iint\limits_D F(x, y, f(x, y))\sqrt{1 + (f'_x)^2 + (f'_y)^2}\,dx\,dy. \tag{2}$$

Example. Compute the surface integral of the first kind

$$\iint\limits_\Omega z\,d\omega$$

over the upper hemisphere of radius R_0 with centre at the origin.

△ The equation of the hemisphere has the form:

$$z = \sqrt{R_0^2 - x^2 - y^2}.$$

Making use of formula (2), we get:

$$\iint\limits_{\Omega} z\, d\omega = \iint\limits_{D} \sqrt{R_0^2 - x^2 - y^2} \cdot \sqrt{\frac{R_0^2}{R_0^2 - x^2 - y^2}}\, dx\, dy$$

$$= R_0 \iint\limits_{D} dx\, dy = \pi R_0^3. \ \blacktriangle$$

Exercises

1. Find the area of the lateral surface of the cone

$$x^2 + y^2 = z^2, \quad 0 \leqslant z \leqslant H.$$

2. Evaluate the surface integrals of the first kind:

(a) $\iint\limits_{\Omega} (x^2 + y^2)\, d\omega$, where Ω is a sphere $x^2 + y^2 + z^2 = a^2$;

(b) $\iint\limits_{\Omega} z\, d\omega$, where Ω is part of the lateral surface of the cone $x^2 + y^2 = z^2$, $0 \leqslant z \leqslant H$;

(c) $\iint\limits_{\Omega} (x + y)\, d\omega$, where Ω is part of the plane $x + y + z = 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

12.7. Surface Integrals of the Second Kind

1. The Definition of the Surface Integral of the Second Kind. Consider the surface Ω in space having a tangent plane at each point. At each point of the surface we choose the direction of the unit vector of the normal $\mathbf{n}$ so that the vector $\mathbf{n}$ is a continuous function of the points of the surface. In this case, the smooth surface is said to be oriented. Let a vector function $\mathbf{a} = \{P(x, y, z), Q(x, y, z), R(x, y, z)\}$ be defined on the surface Ω.

The surface integral of the second kind of the vector function $\mathbf{a} = \mathbf{a}(x, y, z)$ over an oriented surface Ω is defined as the following surface integral of the first kind:

$$\iint\limits_{\Omega} (\mathbf{a}, \mathbf{n})\, d\omega.$$

The surface integral of the second kind is denoted by

$$\iint_{\Omega} (\mathbf{a},\ d\omega) \text{ or } \iint_{\Omega} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy.$$

The first of them is called the vector notation, and the second the coordinate notation. Thus, by definition, the following equalities are valid:

$$\iint_{\Omega} (\mathbf{a},\ d\omega) = \iint_{\Omega} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_{\Omega} (\mathbf{a},\ \mathbf{n})\,d\omega$$

$$= \iint_{\Omega} [P\cos\widehat{(\mathbf{n},\ \mathbf{i})} + Q\cos\widehat{(\mathbf{n},\ \mathbf{j})} + R\cos\widehat{(\mathbf{n},\ \mathbf{k})}]\,d\omega. \quad (1)$$

If the orientation of the surface is changed, that is, instead of the normal vector $\mathbf{n}$, the opposite vector $-\mathbf{n}$ is taken, then the sign of the surface integral of the second kind will be reversed.

The equality

$$\iint_{\Omega} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$= \iint_{\Omega} [P\cos\widehat{(\mathbf{n},\ \mathbf{i})} + Q\cos\widehat{(\mathbf{n},\ \mathbf{j})} + R\cos\widehat{(\mathbf{n},\ \mathbf{k})}]\,d\omega$$

is equivalent to the collection of three equalities

$$\iint_{\Omega} P\,dy\,dz = \iint_{\Omega} P\cos\widehat{(\mathbf{n},\ \mathbf{i})}\,d\omega,$$

$$\iint_{\Omega} Q\,dz\,dx = \iint_{\Omega} Q\cos\widehat{(\mathbf{n},\ \mathbf{j})}\,d\omega,$$

$$\iint_{\Omega} R\,dx\,dy = \iint_{\Omega} R\cos\widehat{(\mathbf{n},\ \mathbf{k})}\,d\omega,$$

which is easily made sure by taking in turn two of the functions P, Q, and R identically equal to zero.

If the surface Ω is given by the equation $z = f(x, y)$, $(x, y) \in D$, where D is the projection of the surface Ω on the xy-plane, then by

formula (2) from 12.6, subsection 3, we find that

$$\begin{aligned}\iint\limits_{\Omega} R\cos(\widehat{\mathbf{n}, \mathbf{k}})\,d\omega &= \iint\limits_{D} R(x, y, f(x, y))\cos(\widehat{\mathbf{n}. \mathbf{k}}) \\ &\quad \times \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy \\ &= \pm\iint\limits_{D} R(x, y, f(x, y))\,\frac{1}{\sqrt{1+(f'_x)^2+(f'_y)^2}} \\ &\quad \times \sqrt{1+(f'_x)^2+(f'_y)^2}\,dx\,dy \\ &= \pm\iint\limits_{D} R(x, y, f(x, y))\,dx\,dy, \qquad (2)\end{aligned}$$

where we have used the expression for $\cos(\widehat{\mathbf{n}, \mathbf{k}})$ from 10.4, subsection 2, and the sign of the integral depends on the choice of orientation of the surface Ω: if an acute angle is formed by the vector $\mathbf{n}$ with the z-axis, then we take the plus sign, and if an obtuse angle, then the minus sign.

Thus, we have found the formula by which the surface integral of the second kind $\iint\limits_{\Omega} R\,dx\,dy$ is reduced to the double integral:

$$\iint\limits_{\Omega} R(x, y, z)\,dx\,dy = \pm\iint\limits_{D} R(x, y, f(x, y))\,dx\,dy. \qquad (3)$$

Similarly, we obtain the following formulas:

$$\iint\limits_{\Omega} P(x, y, z)\,dy\,dz = \pm\iint\limits_{D_2} P(f_1(y, z), y, z)\,dy\,dz, \qquad (4)$$

$$\iint\limits_{\Omega} Q(x, y, z)\,dz\,dx = \pm\iint\limits_{D_3} Q(x, f_2(x, z), z)\,dz\,dx, \qquad (5)$$

where D_2 and D_3 are projections of the surface Ω on the yz- and xz-planes, respectively (it is assumed here that the surface Ω can be given either by the equation $x = f_1(y, z)$, $(y, z) \in D_2$ or by the equation $y = f_2(x, z)$, $(x, z) \in D_3$).

Formula (3) (as also formulas (4) and (5)) can be obtained formally directly from the coordinate notation of the surface integral of the second kind on the left-hand side of equality (3): we have only to replace the variable z by the function $f(x, y)$ which defines the surface Ω and to pass over to the domain D which is the projection of Ω on the xy-plane, and also to choose the sign of the integral which depends on the orientation of the surface Ω. Therefore the coordinate

notation

$$\iint\limits_{\Omega} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

of the surface integral of the second kind is regarded to be successful and is widely spread in mathematical literature. Physical literature most frequently uses the brief vector notation

$$\iint\limits_{\Omega} (\mathbf{a},\ d\boldsymbol{\omega}).$$

Moreover, it prefers, instead of the term "the surface integral of the second kind", to use the term "the flux of the vector **a** through the surface Ω".

Example. Compute the surface integral of the second kind

$$\iint\limits_{\Omega} z\,dx\,dy,$$

where Ω is the upper side of the plane $x + y + z = 1$ located in the first octant (where $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$).

△ The projection of the surface Ω on the xy-plane is represented by the triangle D with vertices at the points (0, 0), (1, 0), and (0, 1). For the upper side of the plane $x + y + z = 1$ the normal vector **n** forms an acute angle with the z-axis. We evaluate the surface integral of the second kind by formula (3):

$$\iint\limits_{\Omega} z\,dx\,dy = \iint\limits_{D} (1-x-y)\,dx\,dy = \int\limits_0^1 dx \int\limits_0^{1-x} (1-x-y)\,dy$$

$$= \int\limits_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_{y=0}^{y=1-x} dx$$

$$= \int\limits_0^1 \left[1 - x - x(1-x) - \frac{(1-x)^2}{2}\right] dx$$

$$= \int\limits_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx = \frac{1}{6}. \ \blacktriangle$$

2. Divergence of a Vector Field. The Gauss-Ostrogradskii Formula. Let in the domain G of a three-dimensional space a vector field $\mathbf{a} = \mathbf{a}(x, y, z)$ be defined:

$$\mathbf{a}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

We shall assume the domain G to be bounded, and its boundary to be piecewise smooth, that is, to have a tangent plane at all points except a finite number of curves and points on the boundary sur-

face Ω. In addition we assume the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, $\frac{\partial R}{\partial z}$ to be continuous in the closed domain G.

Consider the expression equal to the sum of the partial derivatives:

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}. \tag{1}$$

Expression (1) is called the divergence of the vector field $\mathbf{a}(x, y, z)$ and is denoted as follows:

$$\operatorname{div}\mathbf{a}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}. \tag{2}$$

Using the "nabla" vector (Hamiltonian operator, see 12.4), we can write expression (2) as a scalar product of the "vector" $\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$ and the vector $\mathbf{a}$, that is,

$$\operatorname{div}\mathbf{a}=(\nabla, \mathbf{a}). \tag{3}$$

Let us now prove the Gauss-Ostrogradskii formula:

$$\iiint\limits_{G}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz = \iint\limits_{\Omega} P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy. \tag{4}$$

Formula (4) expresses the fact that the triple integral of the divergence of the vector $\mathbf{a}$ over the domain G is equal to the flux of the vector through the boundary of this domain.

The direction of the normal $\mathbf{n}$ to the surface Ω is the direction from the domain towards the outer space with respect to the domain G.

Consider the domain G represented in Fig. 162. Let D denote the projection of the domain G on the xy-plane. In relation to the boundary of the domain Ω, we make a supposition that it is piecewise smooth, and, besides, the straight lines parallel to the z-axis can intersect it at no more than two points. The last supposition implies that the entire boundary of the domain can be made up of two surfaces Ω_1 and Ω_2 having the equations

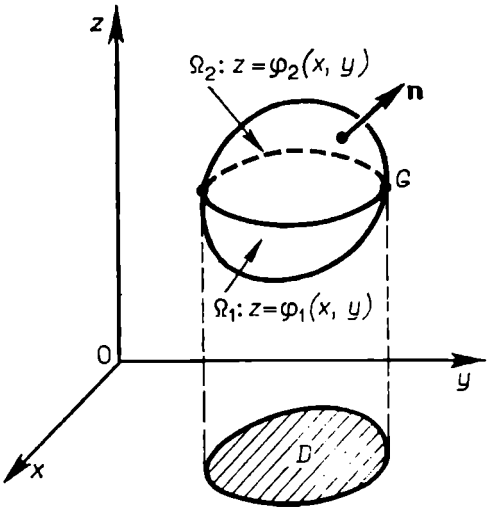

Fig. 162

$$z=\varphi_1(x, y) \quad \text{and} \quad z=\varphi_2(x, y).$$

The surface defined by the equation $z = \varphi_1(x, y)$ bounds the domain G from below (see Fig. 162) and is the locus of "inlet" points for the straight lines passing through the points of the domain D parallel to the z-axis. We denote this surface by Ω_1. The surface defined by the equation $z = \varphi_2(x, y)$ bounds the domain G from above and is the set of the "outlet" points for the straight lines parallel to the z-axis and passing through the points of the domain D in the xy-plane. We denote this surface by Ω_2.

Consider the integral over the domain G with respect to the derivative $\frac{\partial R}{\partial z}$

$$\iiint_G \frac{\partial R}{\partial z}\, dx\, dy\, dz.$$

We compute this integral:

$$\begin{aligned}\iiint_G \frac{\partial R}{\partial z}\, dx\, dy\, dz &= \iint_D dx\, dy \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} \frac{\partial R}{\partial z}\, dz \\ &= \iint_D [R(x, y, \varphi_2(x, y)) - R(x, y, \varphi_1(x, y))]\, dx\, dy \\ &= \iint_{\Omega_2} R(x, y, z)\, dx\, dy + \iint_{\Omega_1} R(x, y, z)\, dx\, dy, \quad (5)\end{aligned}$$

where we have used formula (3) from the preceding subsection and taken into account that the outward normal $\mathbf{n}$ for the surface Ω_2 forms an acute angle with the z-axis and an obtuse angle with the surface Ω_1 (this affects the choice of the sign in formula (3)).

We replace the sum of the integrals over Ω_1 and Ω_2 in expression (5) by one integral over the entire surface Ω bounding the domain G. Thus, we arrive at the formula

$$\iiint_G \frac{\partial R}{\partial z}\, dx\, dy\, dz = \iint_\Omega R(x, y, z)\, dx\, dy. \quad (6)$$

Similarly, computing the integrals

$$\iiint_G \frac{\partial Q}{\partial y}\, dx\, dy\, dz \text{ and } \iiint_G \frac{\partial P}{\partial x}\, dx\, dy\, dz,$$

we get the equalities

$$\iiint_G \frac{\partial Q}{\partial y}\, dx\, dy\, dz = \iint_\Omega Q(x, y, z)\, dz\, dx, \quad (7)$$

$$\iiint_G \frac{\partial P}{\partial x}\, dx\, dy\, dz = \iint_\Omega P(x, y, z)\, dy\, dz. \quad (8)$$

Adding the left-hand and right-hand sides of equalities (6), (7), and (8), we get Gauss-Ostrogradskii formula:

$$\iiint\limits_G \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right) dx\,dy\,dz = \iint\limits_\Omega P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy. \tag{9}$$

3. The Curl of a Vector Field. Stokes' Formula. Let in some domain G of a three-dimensional space a continuously differentiable vector field be given in rectangular Cartesian coordinates:

$$\mathbf{a} = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}.$$

The curl of the vector field $\mathbf{a}$ is defined as the vector equal to

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial x}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}. \tag{1}$$

The curl of the vector field $\mathbf{a}$ is denoted by curl $\mathbf{a}$.

Expression (1) can be written in the form of a symbolic "determinant" of the third order:

$$\operatorname{curl}\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}. \tag{2}$$

The notion of curl is related with the notion of circulation of a vector field round the directed contour C.

The circulation of the vector $\mathbf{a} = \{P, Q, R\}$ round the closed contour C is defined as the magnitude of the line integral of the second kind

$$\int\limits_C P\,dx + Q\,dy + R\,dz.$$

There is a formula, establishing the relation between the curl of a vector field and its circulation which is called Stokes' formula:

$$\iint\limits_\Omega (\operatorname{curl}\mathbf{a},\ \mathbf{n})\,d\omega = \oint\limits_C P\,dx + Q\,dy + R\,dz \tag{3}$$

or

$$\iint\limits_\Omega \left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\widehat{(\mathbf{n},\ \mathbf{i})}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\widehat{(\mathbf{n},\ \mathbf{j})}\right.$$

$$\left.+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\widehat{(\mathbf{n},\ \mathbf{k})}\right] d\omega = \oint\limits_C P\,dx + Q\,dy + R\,dz.$$

Formula (3) expresses the fact that the curl flux of the vector field **a** through the surface Ω with the direction of the normal chosen on it is equal to the circulation of the vector **a** along the boundary of this surface.

The direction of the normal **n** to the surface and the direction of the boundary must be coordinated so that whenever the boundary is traced in the chosen direction, the surface Ω must be kept on the left (see Fig. 163).

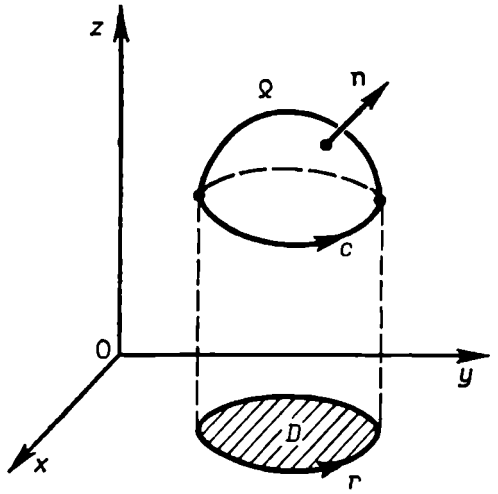

Fig. 163

To derive Stokes' formula, let us assume that the straight lines parallel to the z-axis pierce the surface Ω only at one point (Fig. 163), that is, Ω is defined by the equation $z = f(x, y)$.

The projection of the boundary C on the xy-plane yields the boundary Γ of the domain D. We choose the direction of tracing the boundary Γ of the domain D in conformity with the direction of tracing the boundary C. We choose the direction of the normal **n** to the given surface so that the normal forms an acute angle with the z-axis. Then the direction cosines of the normal to the surface Ω are found by the formulas (see 10.4, subsection 2)

$$\cos\widehat{(\mathbf{n}, \mathbf{i})} = \frac{-f'_x}{\sqrt{1+(f'_x)^2+(f'_y)^2}},$$

$$\cos\widehat{(\mathbf{n}, \mathbf{j})} = \frac{-f'_y}{\sqrt{1+(f'_x)^2+(f'_y)^2}}, \qquad (4)$$

$$\cos\widehat{(\mathbf{n}, \mathbf{k})} = \frac{1}{\sqrt{1+(f'_x)^2+(f'_y)^2}}.$$

Formula (4) implies the equalities

$$f'_x \cos\widehat{(\mathbf{n}, \mathbf{k})} = -\cos\widehat{(\mathbf{n}, \mathbf{i})}, \quad f'_y \cos\widehat{(\mathbf{n}, \mathbf{k})} = -\cos\widehat{(\mathbf{n}, \mathbf{j})}. \qquad (5)$$

Let $P(x, y, z)$ be a function defined on the surface Ω and in some neighbourhood. This function is assumed to be continuously differentiable. Consider the line integral of the second kind

$$\oint_C P(x, y, z)\,dx = \oint_\Gamma P(x, y, f(x, y))\,dx. \qquad (6)$$

We apply the Riemann-Green formula (see 12.5, subsection 3) to the integral on the right-hand side of equality (6). Then we get

$$\oint_C P(x, y, z)\,dx = \oint_L P(x, y, f(x, y))\,dx$$
$$= -\iint_D \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\cdot\frac{\partial f}{\partial y}\right) dx\,dy. \qquad (7)$$

Here P is a composite function of x and y, and we have taken this into account when computing the derivative of P with respect to y.

Making use of formula (2) in subsection 1 and passing over in formula (7) from the integral over the domain D to the integral over the surface Ω, we get

$$\oint_C P(x, y, z)\,dx = -\iint_D \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\cdot\frac{\partial f}{\partial y}\right) dx\,dy$$
$$= -\iint_\Omega \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\cdot\frac{\partial f}{\partial y}\right) \cos(\widehat{\mathbf{n}, \mathbf{k}})\,d\omega.$$

Using formula (5), we obtain

$$\oint_C P\,dx = \iint_\Omega \left(\frac{\partial P}{\partial z}\cos(\widehat{\mathbf{n}, \mathbf{j}}) - \frac{\partial P}{\partial y}\cos(\widehat{\mathbf{n}, \mathbf{k}})\right) d\omega. \qquad (8)$$

Similarly, for the integrals

$$\oint_C Q\,dy \quad \text{and} \quad \oint_C R\,dz,$$

we get the equalities

$$\oint_C Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x}\cos(\widehat{\mathbf{n}, \mathbf{k}}) - \frac{\partial Q}{\partial z}\cos(\widehat{\mathbf{n}, \mathbf{i}})\right) d\omega. \qquad (9)$$

$$\oint_C R\,dz = \iint_\Omega \left(\frac{\partial R}{\partial y}\cos(\widehat{\mathbf{n}, \mathbf{i}}) - \frac{\partial R}{\partial x}\cos(\widehat{\mathbf{n}, \mathbf{j}})\right) d\omega. \qquad (10)$$

Adding together equalities (8), (9), and (10), we arrive at Stokes' formula:

$$\oint_C P\,dx + Q\,dy + R\,dz = \iint_\Omega \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos(\widehat{\mathbf{n}, \mathbf{i}})\right.$$
$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos(\widehat{\mathbf{n}, \mathbf{j}})$$
$$\left. + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos(\widehat{\mathbf{n}, \mathbf{k}})\right] d\omega.$$

And this is the vector form of Stokes' formula:

$$\int_C (\mathbf{a},\ d\mathbf{r}) = \iint_\Omega (\operatorname{curl}(\mathbf{a},\ \mathbf{n})\, d\omega.$$

Exercises

1. Compute the flux of the vector field $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the surface of the sphere $x^2 + y^2 + z^2 = 1$.
2. Find the divergence of the vector $\mathbf{a} = r\mathbf{r}$, where $\mathbf{r} = \{x, y, z\}$.
3. Find the divergence of the vector $\mathbf{a} = \{x^2, y^2, z^2\}$.
4. Find div (grad u), where $u = x^2 + y^2 + z^2$.
5. Compute curl $\mathbf{a}$ if: (a) $\mathbf{a} = x^2\mathbf{i} - y^2\mathbf{j} + z\mathbf{k}$; (b) $\mathbf{a} = \frac{\mathbf{r}}{r^2}$; (c) $\mathbf{a} = \mathbf{r}$.
6. Making use of the Gauss-Ostrogradskii formula, evaluate the surface integral

$$\iint_\Omega x\, dy\, dz + y\, dx\, dz + z\, dx\, dy,$$

where Ω is the external side of the pyramid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.
7. Compute curl (grad u), where

(a) $u = x^2 + y^2 + z^2$; (b) $u = \sqrt{x^2 + y^2 + z^2}$;

(c) $u = \frac{1}{r}$, where $r = |\mathbf{r}|$.

8. Compute div (curl $\mathbf{a}$) if: (a) $\mathbf{a} = \frac{\mathbf{r}}{r}$; (b) $\mathbf{a} = r\mathbf{r}$; (c) $\mathbf{a} = \mathbf{r}$.

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HIGHER MATHEMATICS

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